A logarithmic approach to spaces of multi-scale differentials

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Motivation

$g \geq 0$, $\mu = (m_1, \dots, m_n) \in \mathbb{Z}^n$
w/ $|\mu| = \sum m_i = 2g-2$

$$\mathcal{H}_g(\mu) = \left\{ (C, P_1, \dots, P_n, \eta) \;\middle|\; \begin{array}{l} C \text{ smooth gen. } g \text{ curve}, \\ P_i \in C \text{ pairwise distinct}, \\ \eta \text{ meromorphic differential} \\ \text{on } C \text{ w/ } \operatorname{div}(\eta) = \sum m_i P_i \end{array} \right\} \Big/ \sim$$

η has zeros/poles precisely at P_i, orders m_i

$(C, P_1, \dots, P_n, \eta) \sim (C, P_1, \dots, P_n, \lambda\eta)$, $\lambda \in \mathbb{C}^*$

Guiding question

How to define a nice compactification of $\mathcal{H}_g(\mu)$?

I) Moduli spaces of multi-scale differentials

Q Ambient space for $\mathcal{H}_g(\mu)$?

Write $\mu = \mu^+ - \mu^-$ for $\mu^+, \mu^- \in \mathbb{Z}^n_{\geq 0}$.

e.g. $(5,3,-4) = (5,3,0) - (0,0,4)$

$$\begin{tikzcd}
\mathcal{C} \arrow[d, "\pi"'] & \omega_\pi\left(\sum_{i=1}^n \mu_i^- P_i\right) \arrow[r, squiggly] & \mathbb{E}_{g,\mu^-} = \pi_* \omega_\pi\left(\sum_{i=1}^n \mu_i^- P_i\right) \arrow[d] \\
\overline{\mathcal{M}}_{g,n} \arrow[u, bend right, "P_i"'] & & \overline{\mathcal{M}}_{g,n}
\end{tikzcd}$$

twisted Hodge bundle

C smooth
⇝ sections on $\pi^{-1}([C])$
= merom. diff. on C with poles of orders at most $|m_i|$ at pts P_i with $m_i < 0$

$$\begin{tikzcd}
\mathcal{H}_g(\mu) \arrow[r, hook, "\text{loc. closed immers.}"] \arrow[rd, hook, "\text{loc. closed immers.}"'] & \mathbb{P}\mathbb{E}_{g,\mu^-} \arrow[d] & \supseteq \overline{\mathrm{IVC}}_g(\mu) \\
& \overline{\mathcal{M}}_{g,n} & \supseteq \overline{\mathcal{H}}_g(\mu)
\end{tikzcd}$$

closure of $\mathcal{H}_g(\mu)$

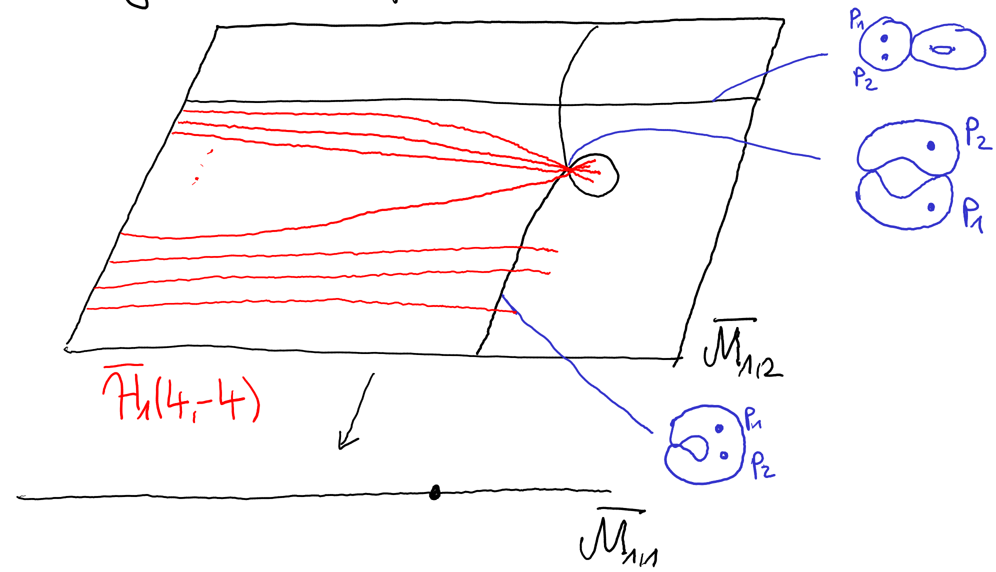
incidence variety compactification

Deligne-Mumford compactification

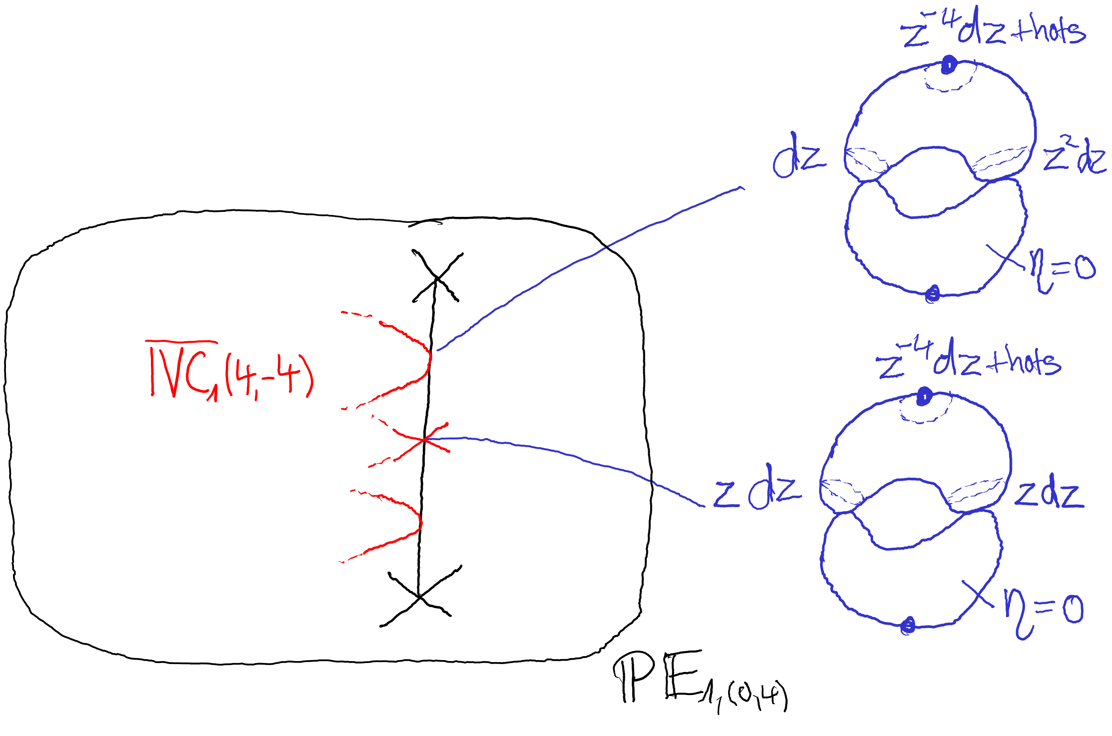
Problem $\overline{\mathrm{IVC}}_g(\mu)$ and $\overline{\mathcal{H}}_g(\mu)$ are not smooth

Exa $g=1$, $\mu=(4,-4)$

Deligne-Mumford compactification



Incidence variety compactification



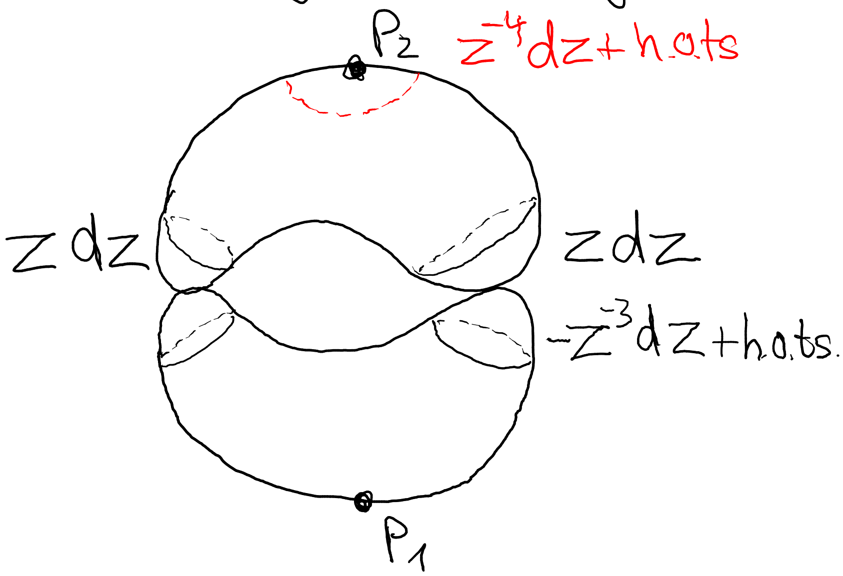
Upshot

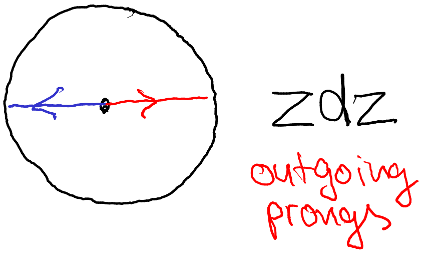
- $\overline{IVC}_g(\mu)$ less singular than $\overline{\mathcal{H}}_g(\mu)$

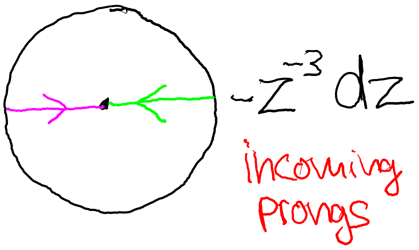
  ⇝ remembering the differentials helps
- Still some remaining singularities

  ⇝ need to remember more data from degeneration

Prong matchings

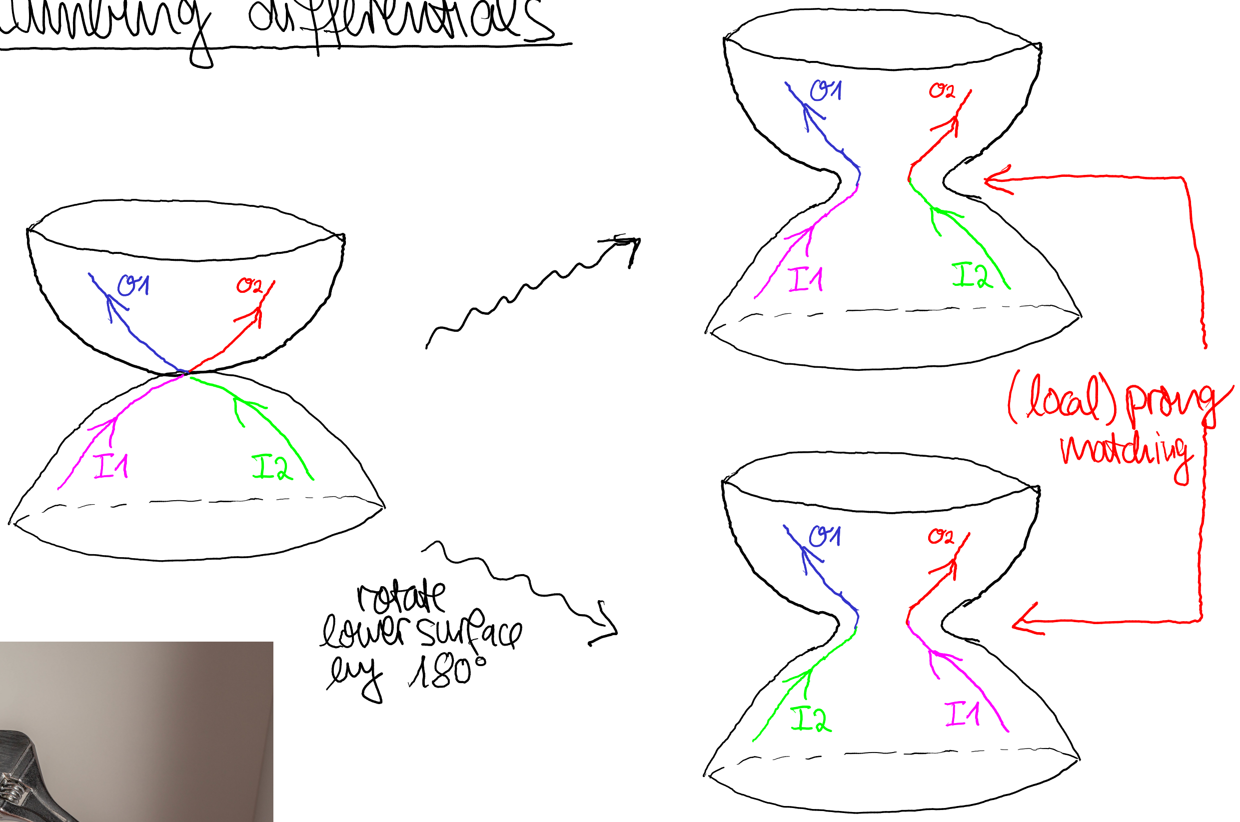






prong : geodesic γ through (preimage of) node s.t. $\eta\left(\frac{\partial\gamma}{\partial t}\right) \in \mathbb{R}^+$ ⇝ horizontal geodesic

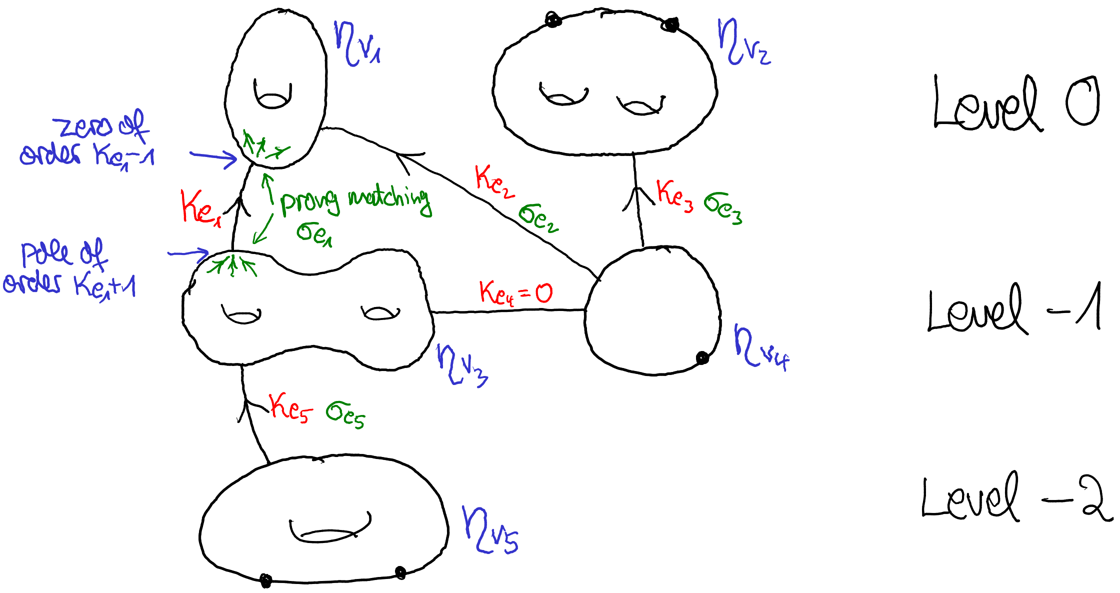
Plumbing differentials




[1]

Def (Bainbridge-Chen-Gendron-Grushevsky-Möller)

The space $GMS_{g,\mu}$ of generalized multi-scale differentials parameterizes data



- $(C, p_1, \dots, p_n)$ stable curve of genus g
- level structure $\ell: V(\Gamma) \to \{0,-1,\dots,-N\}$ on stable graph $\Gamma = \Gamma(C)$
- twists $\kappa_e \in \mathbb{Z}_{\geq 0}$ on edges ; $\kappa_e = 0 \Leftrightarrow$ e connects vert. on same level
- twisted differentials η_v on components C_v of C
  - → zero/pole of order m_i at p_i
  - → zero of order $\kappa_e - 1$ at top of vertical edge
    pole of order $\kappa_e + 1$ at bottom of vertical edge
    simple poles w/ opposite residues at ends of horizontal edge
- local prong matching σ_e at vertical edges e

Notion of isomorphism

levelwise rescaling* of differentials η_v on levels $-1, -2, \dots, -N$

all η_v on same level get scaled by same constant

* Scaling of η_v rotates the prongs w. speed depending on κ_e

⇝ group responsible for rescaling is finite cover

$$T_\Gamma^{(s)} \longrightarrow (\mathbb{C}^*)^N$$

called (simple) level rotation torus

Variants of the definition of $GMS_{g,\mu}$

- $\mathbb{P}GMS_{g,\mu}$: also simultaneously rescale differentials η_v on level 0

$$\begin{tikzcd} \rightsquigarrow & \mathbb{P}GMS_{g,\mu} \arrow[r, "\uparrow \text{ set differential to zero on levels } -1,-2,\dots,-N"'] & \mathbb{P}\mathbb{E}_{g,\mu^-} & \leftarrow \text{projectivized Hodge bundle} \end{tikzcd}$$

- $\mathbb{P}MS_{g,\mu}$: impose additional global residue condition

$$\begin{tikzcd} \rightsquigarrow & \mathbb{P}MS_{g,\mu} \arrow[r, hook, "\text{closed}", "\text{substack}"'] \arrow[rrd, two heads] & \mathbb{P}GMS_{g,\mu} \arrow[r] & \mathbb{P}\mathbb{E}_{g,\mu^-} \arrow[d, phantom, sloped, "\supseteq"] \\ \text{projectivized multi-scale differentials} \nearrow & & & \overline{IVC}_{g,\mu} \end{tikzcd}$$

> **Thm** (BCGGM)
> The stack $\mathbb{P}MS_{g,\mu}$ is a smooth, proper DM-stack, containing $\mathcal{H}_g(\mu)$ as a dense open substack.

Summary

- by remembering curve C, level graph Γ, twists κ_e, differentials η_v and prong matchings σ_e up to levelwise scaling we obtained a stack $\mathbb{P}GMS_{g,\mu}$ of generalized multi-scale diff.
- closed substack $\mathbb{P}MS_{g,\mu}$ is smooth compactification of $\mathcal{H}_g(\mu)$

Question Conceptual explanation for this definition?

II) Logarithmic rubber differentials

Basic idea Reformulate existence of differential η in terms of line bundles

C smooth curve

$$\rightsquigarrow \left(\begin{array}{l} \exists \text{ merom. diff. } \eta \text{ on } C \\ \text{w/ } \mathrm{div}(\eta) = \sum_i m_i p_i \end{array} \right) \Longleftrightarrow \omega_C \cong \mathcal{O}_C\left(\sum_i m_i p_i\right)$$

$$\Longleftrightarrow \omega_C\left(-\sum_i m_i p_i\right) \underset{\uparrow}{\cong} \mathcal{O}_C$$

equality in space of line bundles on C

$$\mathrm{Pic}_g = \left\{ (C, \mathcal{L}) \;\middle|\; \begin{array}{l} C \text{ nodal curve, a. genus } g, \\ \mathcal{L}/C \text{ line bundle} \end{array} \right\}$$

universal Picard stack

$$\begin{array}{l} \cup\!\!\!| \\ e = \{ (C, \mathcal{L}) \mid C \text{ smooth, } \mathcal{L} = \mathcal{O}_C \} \end{array}$$

$$\begin{tikzcd} \mathcal{H}_g(\mu) \arrow[r, hook, "\circ"] \arrow[d] & \widetilde{\mathcal{H}}_g(\mu) \arrow[r, hook] \arrow[d] & \overline{\mathcal{M}}_{g,n} \arrow[d, "{\begin{array}{c}(C,p_1,\ldots,p_n)\\ \downarrow \\ (C,\omega_C(-\sum m_i p_i))\end{array}}"] \\ e \arrow[r, hook, "\circ"] & \overline{e} \arrow[r, hook] & \mathrm{Pic}_g \end{tikzcd}$$

$\uparrow$ closure of $e \subseteq \mathrm{Pic}_g$

$\rightsquigarrow \widetilde{\mathcal{H}}_g(\mu)$ = moduli space of twisted differentials [Farkas-Pandharipande]

$\rightsquigarrow$ this contains the DM compactification $\overline{\mathcal{H}}_g(\mu)$ as closed substack (in fact: union of components)

$\rightsquigarrow$ still singular

More advanced idea

Find sophisticated space

$$\begin{tikzcd} \boxed{???} \arrow[r, two heads] & \overline{e} \arrow[r, hook] & \mathrm{Pic}_g \end{tikzcd}$$

and form same fibre diagram.

The space Rub

Def ([Marcus-Wise], Rub for log geometers)
Rub is the stack (over f.s. log schemes) with objects

$$(\pi : C \longrightarrow B\ ,\ \beta : C \longrightarrow \mathbb{G}_{m,B}^{trop})$$

with C/B a log curve, satisfying

- the image of β is fibrewise totally ordered, with largest element 0
- writing R for the stack obtained from $\mathbb{G}_m^{trop}$ by subdividing at the image of β, we have that $C \times_{\mathbb{G}_m^{trop}} R$ is a log curve.

Log geometry at work [2]

⇝ can use $Rub \twoheadrightarrow \overline{e} \longleftrightarrow Pic_g$ above

Rub for non-log geometers

Crash course Log geometry

Log scheme $(X, M_X, \alpha : M_X \longrightarrow (\mathcal{O}_X, \cdot))$

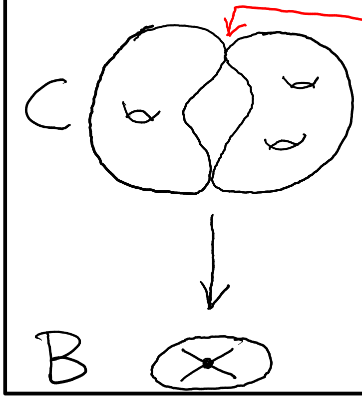
scheme ↗ sheaf of monoids on X ↑ ↖ monoid morphism with $\alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow[\sim]{\alpha} \mathcal{O}_X^\times$

We obtain

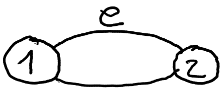
$$1 \longrightarrow \mathcal{O}_X^\times \xrightarrow{\alpha^{-1}} M_X \longrightarrow \overline{M}_X \longrightarrow 0$$

↑ ghost sheaf

$C \longrightarrow B$ log curve ⇜ proper, integral, saturated, log smooth, geom. fibers = reduced, pure dim 1



$\delta_e \in \overline{M}_B$ such that $\alpha(\delta_e) \in \mathcal{O}_B/\mathcal{O}_B^\times$ is a "smoothing parameter" for node assoc. to e



What are the families of Rub over $B = \operatorname{Spec}\mathbb{C}$?

in paper : (B, b) nuclear log scheme

$$\mathrm{Rub}(B) = \left\{ \left(\begin{matrix} C \\ \downarrow \\ B \end{matrix} \ \underset{\Gamma = \Gamma(C)}{\text{log curve}}, \ \beta : V(\Gamma) \longrightarrow \overline{M}_B^{gp} \right) \ \middle| \ \circledast \right\}$$

piecewise linear function

groupification of stalk $\overline{M}_B$

$\circledast$

(1) For all $e \in E(\Gamma)$ we have divisibility condition

$$\delta_e \mid \beta(v_2) - \beta(v_1) \in \overline{M}_B^{gp}$$

e.g. $\overline{M}_B = \mathbb{N}^2$

$\rightsquigarrow \overline{M}_B^{gp} = \mathbb{Z}^2$

$\beta(v_1) = (-4, -7)$

$\beta(v_2) = (-2, -3)$

$\delta_e = (1, 2)$

$\rightsquigarrow \beta(v_2) - \beta(v_1) = 2 \cdot \delta_e$

Define $K_e \in \mathbb{Z}_{\geq 0}$ by

$$\beta(v_2) - \beta(v_1) = \pm K_e \cdot \delta_e$$

(2) The image of β in $\overline{M}_B^{gp}$ is totally ordered with maximal element $0 \in \overline{M}_B^{gp}$.

e.g. $\beta(v_1) = (-4, -7)$, $\beta(v_2) = (-2, -3)$ $\Rightarrow$ $\beta(v_1) \leq \beta(v_2)$ since $\beta(v_2) - \beta(v_1) = (2, 4) \in \underset{\underset{\overline{M}_B}{\parallel}}{\mathbb{N}^2}$

technical condition

(3) For all $\underset{\underset{V(\Gamma)}{\cap}}{v}$

$\cap$
$E(\Gamma)$

s.t. $$\beta(v_2) > \gamma = \beta(v) > \beta(v_1)$$ we have $$\frac{\gamma - \beta(v_1)}{K_e} \in \overline{M}_B \, .$$

$\beta : V(\Gamma) \longrightarrow \overline{M}_B^{gp}$ satisf. (1) $\iff$ $\beta \in H^0(C, \overline{M}_C^{gp})$ $\iff$ $\beta : C \to \mathbb{G}_m^{trop}$

$$\begin{tikzcd}
\rightsquigarrow \quad 1 \arrow[r] & \mathcal{O}_C^\times \arrow[r, "\alpha"] & M_C^{gp} \arrow[r, "q"] & \overline{M}_C^{gp} \arrow[r] & 0 \\
& & q^{-1}(\beta) \arrow[r, dashed, no head] & \beta \arrow[u, phantom, sloped, "\in"] & \\
& & \mathcal{O}_C^\times\text{-torsor} \arrow[u] & \iff \mathcal{O}_C(\beta) &
\end{tikzcd}$$

line bundle on C

The bundle $\mathcal{O}_C(\beta)$ satisfies:

$$\mathcal{O}_C(\beta)\Big|_{C_v} = \mathcal{O}_{C_v}\Big(\sum_{e \in E_v^-} \kappa_e q_e + \sum_{e \in E_v^+} (-\kappa_e) q_e\Big)$$

$$\begin{tikzcd}
\rightsquigarrow \quad \mathrm{Rub} \arrow[r, "aj"] & \mathrm{Pic}_g \\
(C/B, \beta) \arrow[r, mapsto] & (C/B, \mathcal{O}_C(\beta))
\end{tikzcd}$$

Then

$$\begin{tikzcd}
\mathrm{Rub} \arrow[r, "\mathbb{G}_m\text{-bundle}"] \arrow[rrr, bend left, "aj"] & \mathrm{Rub}^{abs} \arrow[r, "\text{proper}", "\text{birational}"'] & \widetilde{\mathcal{C}} & \subseteq \mathrm{Pic}_g
\end{tikzcd}$$

<u>Def</u> Define the stack $\mathrm{Rub}_{\mathcal{L},\mu}$ as the fibre product

$$\begin{tikzcd}
\mathrm{Rub}_{\mathcal{L},\mu} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\square"] & \overline{\mathcal{M}}_{g,n} \arrow[d, "\mathcal{L} = \omega(-\sum m_i p_i)"] \\
\mathrm{Rub} \arrow[r, "aj"] & \mathrm{Pic}_g
\end{tikzcd}$$

Theorem (Chen-Grushevsky-Holmes-Möller-S.)
There exists an isomorphism of algebraic stacks

$$\mathrm{Rub}_{\mathcal{L},\mu} \xrightarrow{\sim} \mathrm{GMS}_{g,\mu}$$

over $\overline{\mathcal{M}}_{g,n}$.

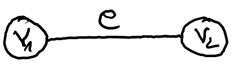
Sketch of comparison

Elements of $\mathrm{Rub}_{\mathcal{L},\mu}$: $\left(C/B,\ \beta : V(\Gamma) \to \overline{M}_B^{gp},\ \mathcal{O}_C(\beta) \xrightarrow[\sim]{\varphi} \omega_C(-\Sigma m_i p_i)\right)$

$\wr$

Reconstruct data of a generalized multi-scale differential.

- Curve $C = C$ ✓
- Level function $\ell : V(\Gamma) \xrightarrow{\beta} \underset{\overline{M}_B^{gp}}{\underset{\cap}{\mathrm{im}(\beta)}} \overset{\mathrm{im}(\beta)\ \text{tot. ordered}}{\cong} \{0, -1, \ldots, -N\}$
- Twists $\kappa_e \in \mathbb{Z}_{\geq 0}$ already appear in def. of Rub



$\rightsquigarrow \kappa_e = \left| \frac{\beta(v_2) - \beta(v_1)}{\delta_e} \right|$

- Twisted differentials η_v & Prong matchings σ_e

For these we first need to choose some extra data

Def A log splitting $\widetilde{\Psi}$ is a section

$$\begin{tikzcd} M_B^{gp} \arrow[r, "q"] & \overline{M}_B^{gp} \arrow[l, bend right, "\widetilde{\Psi}"'] \end{tikzcd}.$$

Then

$m \in \overline{M}_B^{gp} \rightsquigarrow \widetilde{\Psi}(m)$ section of $\mathcal{O}(m) \cong q^{-1}(m)$

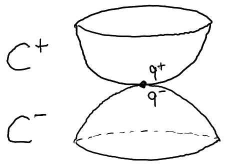
very roughly

$\beta(v) \rightsquigarrow \eta_v = \widetilde{\Psi}(\beta(v))$ section of $\mathcal{O}(\beta(v))|_{C_v^{sm}} \overset{\varphi}{\cong} \omega_{C_v^{sm}}$

twisted differential

$\delta_e \rightsquigarrow \sigma_e = \widetilde{\Psi}(\delta_e)$ section of $\mathcal{O}(\delta_e) \cong T_{q^+}^* C^+ \otimes T_{q^-}^* C^-$

prong matching



Finally:

$\left\{ \begin{array}{c} \text{log splittings} \\ \widetilde{\Psi} : \overline{M}_B^{gp} \to M_B^{gp} \end{array} \right\}$ is torsor under the simple level rotation torus T_Γ^s from [BCGGM]

Action of T_Γ^s compatible with construction of η_v and σ_e above. ☐

Applications & Open problems

- Obtain the smooth compactification $\mathbb{P}\mathcal{MS}_{g,\mu}$ of $\mathcal{H}_g(\mu)$ as an explicit blow-up of
  - → normalization of $\overline{IVC}_{g,\mu}$ $(g \geq 0)$
  - → the space $\overline{\mathcal{M}}_{0,n}$ $(g=0)$

- Have a map $\mathbb{P}\mathrm{Rub}_{\mathcal{L},\mu} \xrightarrow{F} \mathbb{P}\mathbb{E}_{g,\mu^-}$ to project. Hodge bundle

Def (Hodge DR cycle)

$$\widetilde{DR}_g(A) = F_*[\mathbb{P}\mathrm{Rub}_{\mathcal{L},\mu}]^{\mathrm{vir}} \in CH_{2g-3+n}(\mathbb{P}\mathbb{E}_{g,\mu^-})$$

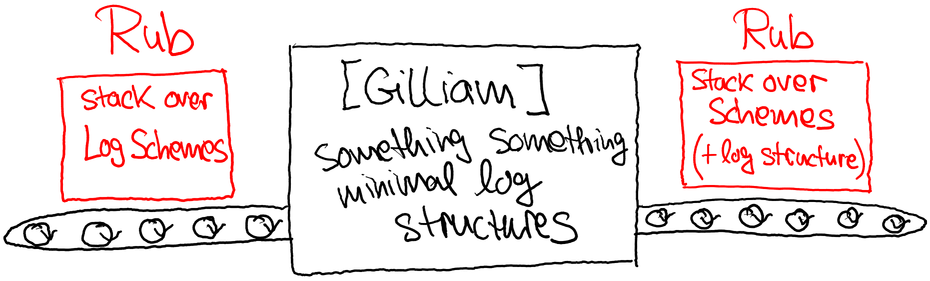
$A=(m_1+1, \ldots, m_n+1)$ log convention

Conjecture For $\mathbb{P}\mathbb{E}_{g,\mu^-} \xrightarrow{p} \overline{\mathcal{M}}_{g,n}$ and $\eta = c_1(\mathcal{O}_{\mathbb{P}\mathbb{E}_{g,\mu}}(1))$:

$$p_*(\overline{DR}_g(A) \cdot \eta^u) = [r^u]\, Ch_{g,A}^{k=1,r,g+u} \in CH^{g+u}(\overline{\mathcal{M}}_{g,n})$$

take coeff. of r^u Chiodo class

- Functor of points for Rub over Schemes ?



Thank you for your attention!

[3] Plumbers vs. Loggers

Image sources:
[1] Plumber with adjustable wrench repairing pipes - Marco Verch Professional Photographer
URL: https://www.flickr.com/photos/30478819@N08/51110597526
[2] DALLE-2, prompt "A high quality drawing of a futuristic robot, cutting down trees in a forest"
[3] DALLE-2, prompt "Fight between a plumber with a wrench and a woodcutter with an axe, high quality digital art"

Appendix

More details from comparison of $Rub_{\mathcal{L},\mu}$ to $GMS_{g,\mu}$

Def A log splitting $\widetilde{\Psi}$ is a section

$$\begin{tikzcd} M_B^{gp} \arrow[r, "q"] & \overline{M}_B^{gp} \arrow[l, bend right, "\widetilde{\varphi}"'] \end{tikzcd} .$$

On C_v^{sm}: $\leftarrow$ smooth, non-marking pts. of C_v

$$\beta \in H^0(C, \overline{M}_C^{gp}) \xrightarrow[\text{section}]{\text{restricts to constant}} \beta(v_i) \in H^0(C_v^{sm}, \overline{M}_C^{gp}) = \overline{M}_B^{gp}$$

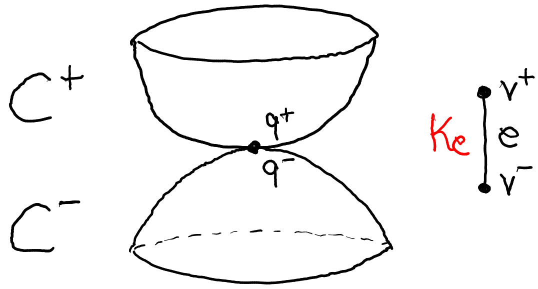
$\Rightarrow$ bundle $\mathcal{O}_C(\beta)|_{C_v^{sm}}$ obtained from $\mathcal{O}_{C_v^{sm}}^{\times}$-torsor $q^{-1}(\beta(v_i))$

has section $\widetilde{\Psi}(\beta(v_i))$

Then

$$\begin{tikzcd} \mathcal{O}_C(\beta)|_{C_v^{sm}} \arrow[r, "\varphi", "\sim"'] & \omega_{C_v^{sm}} & \\ \widetilde{\Psi}(\beta(v_i)) \arrow[r, mapsto] & \eta_v & \text{twisted differential} \arrow[l] \end{tikzcd}$$

Prong matching



$$\mathcal{N}_e := T_{q^+}C^+ \otimes T_{q^-}C^-$$

$$\wr$$

$$\mathcal{I}_e := \eta_{v^+} \otimes \eta_{v^-} \underset{\nwarrow \text{check!}}{\in} \mathcal{N}_e^{\kappa_e}$$

Fact 1 A prong matching at q is an element $\sigma_e \in \mathcal{N}_e^\vee$ s.t.

$$\sigma_e^{\kappa_e}(\mathcal{I}_e) = 1$$

Sanity check: $\exists$ κ_e choices for σ_e

Fact 2 For $\delta_e \in \overline{M}_B$ there is a natural isomorphism

$$\mathcal{O}_B(\delta_e) \cong \mathcal{N}_e^\vee.$$

log splitting : $\widetilde{\Psi}(\delta_e)$ gives section of $\mathcal{O}_B(\delta_e)$ w/ $\widetilde{\Psi}(\delta_e)(\mathcal{I}_e) = 1$.

$\rightsquigarrow \widetilde{\Psi}(\delta_e) =$ prong matching σ_e

Finally $\left\{ \begin{matrix} \text{log splittings} \\ \widetilde{\Psi} : \overline{M}_B^{gp} \to M_B \end{matrix} \right\}$ is torsor under the simple level rotation torus T_Γ^s from BCGGM.

Action of T_Γ^s compatible w/ construction of η_v and $\mathcal{I}_e$ above $\square$

Cartoon summary

	$\mathrm{Rub}_{\mathcal{L},\mu}$	$\mathrm{GMS}_{g,\mu}$	
	$\beta : V(\Gamma) \to \overline{M}_B^{gp}$	level structure on Γ, twists κ_e	
choice of log splitting $\circlearrowleft$ $\{$	log splitting $\widetilde{\Psi}$ & $\mathcal{O}_C(\beta) \xrightarrow[\sim]{\varphi} \omega_C(-\Sigma m_i p_i)$	twisted differentials η_v prong matchings σ_e	$\}$ $\circlearrowleft$ simple level rotation torus T_Γ^s

Applications & related topics

1| Blowup descriptions of spaces of multiscale differentials

Thm (CGHMS)

There exists an explicit iterated blowup of boundary strata

$$\widehat{\overline{\mathcal{M}}}{}^{\mu}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n} \qquad (\text{isom. over } \mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n})$$

Such that

$$\overline{\mathcal{H}_g(\mu)}^{\widehat{\overline{\mathcal{M}}}{}^{\mu}_{g,n}} = \mathbb{P}\mathrm{MS}^{\text{coarse}}_{g,\mu}$$

relative coarse space over $\overline{\mathcal{M}}_{g,n}$

[BCGGM] multiscale space

In particular $g = 0 \rightsquigarrow \mathcal{H}_g(\mu) = \mathcal{M}_{0,n}$, So that $\mathbb{P}\mathrm{MS}^{\text{coarse}}_{g,\mu} = \widehat{\overline{\mathcal{M}}}{}^{\mu}_{g,n}$ is a blowup of $\overline{\mathcal{M}}_{0,n}$.

Remarks

- Replacing "iterated blowup of bdry" by "log modification", one can remove the word coarse above.
- Explicit description: log modification $\longleftrightarrow$ subdivision of $M^{trop}_{g,n} =: \Sigma_{g,n}$

  For DR experts: can construct such subdivision $\Sigma^{\theta}_{g,n}$ using stability conditions θ on line bundles; take the one from [Holmes-Molcho-Pandharipande-Pixton-S.] supporting the log double ramification cycle $\log DR_g(A)$

  $A = (m_1+1, \ldots, m_n+1)$.
- Similar Can write $\mathbb{P}\mathrm{MS}^{\text{coarse}}_{g,\mu}$ as explicit blowup of normalization of $\overline{\mathrm{IVC}}_{g,\mu}$

2) The Hodge double ramification cycle

$A = (a_1, \dots, a_n) \in \mathbb{Z}^n$ with $|A| = \sum_i a_i = k(2g-2+n)$ $(k \in \mathbb{Z})$

$\mathcal{L}_A = \omega^{\otimes k}(-\sum (a_i - k) p_i)$ on $C \xrightarrow{\pi} \overline{\mathcal{M}}_{g,n}$

$$\begin{tikzcd} \mathrm{Rub}_{\mathcal{L}_A} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{g,n} \arrow[d, "aj_{\mathcal{L}_A} : \mathcal{L} = \mathcal{L}_A"] \\ \mathrm{Rub} \arrow[r, "aj"] & \mathcal{P}ic_g \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Rub}_{\mathcal{L}_A} \ni (C/B, \beta \in H^0(C, \overline{M}_C^{gp}), \underset{\underset{\mathbb{G}_m}{\circlearrowleft}}{\mathcal{O}_C(\beta)} \xrightarrow[\sim]{\varphi} \mathcal{L}_A) \arrow[d, squiggly] \\ \mathbb{P}\mathrm{Rub}_{\mathcal{L}_A} = \mathrm{Rub}_{\mathcal{L}_A} / \mathbb{G}_m \end{tikzcd}$$

$[\mathbb{P}\mathrm{Rub}_{\mathcal{L}_A}]^{vir} := aj_{\mathcal{L}_A}^! [\mathrm{Rub}]$ virtual class

For $k \geq 1$

$$\begin{tikzcd} \mathbb{E}_{g,A^-}^k = \pi_* \omega_\pi^{\otimes k}(-\sum_{i : a_i < k} (a_i - k) p_i) \arrow[d, "\text{vector bundle}"] \\ \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

twisted k-th Hodge bdle.

$\rightsquigarrow$
$$\begin{tikzcd} \mathbb{P}\mathrm{Rub}_{\mathcal{L}_A} \arrow[r, "F"] \arrow[dr, "p"'] & \mathbb{P}\mathbb{E}_{g,A^-}^k \arrow[d, "q"] \\ & \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

Def (Hodge DR cycle)

$$\widetilde{DR}_{g,A}^k := F_* [\mathbb{P}\mathrm{Rub}_{\mathcal{L}_A}]^{vir} \in CH^*(\mathbb{P}\mathbb{E}_{g,A^-}^k)$$

Projective bundle formula for q

$\Rightarrow$ $\widetilde{DR}_{g,A}^k$ uniquely determined by the cycles

$$q_*(\widetilde{DR}_{g,A}^k \cdot H^u) \in CH^{g+u}(\overline{\mathcal{M}}_{g,n}), \quad H = c_1(\mathcal{O}(1)) \text{ on } \mathbb{P}\mathbb{E}_{g,A^-}^k$$

$$\parallel$$

$$p_*([\mathbb{P}\mathrm{Rub}_{\mathcal{L}_A}]^{vir} \cdot \eta^u) \qquad \eta = F^* H.$$

For K=0

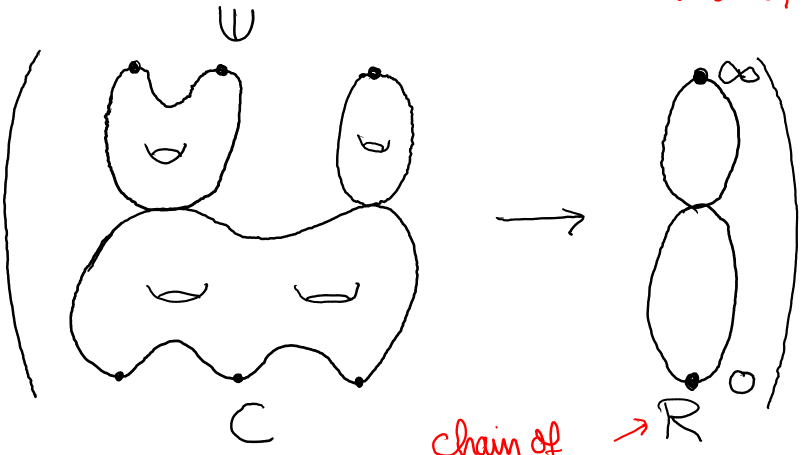
Story above does not quite work:

$E^0_{g,A^-} = \pi_* \mathcal{O}_C\left(-\sum_{i:a_i<0} a_i p_i\right)$ not a vector bundle!

Instead [Bae-Holmes-Pandharipande-S.-Schwarz] shows

$$\mathbb{P}\mathrm{Rub}_{\mathcal{L}_A} \cong \overline{\mathcal{M}}_{g,A}(\mathbb{P}^1, 0, \infty)^\sim \leftarrow$$

moduli space of rubber maps to $\mathbb{P}^1$ relative to $0,\infty \in \mathbb{P}^1$



chain of rat'l curves

This space carries a natural class

$$\eta = \Psi_\infty = c_1(T^*_\infty R)$$

↖ cotangent line bundle at $\infty \in R$

For arbitrary K≥0

What is $p_*([\mathbb{P}\mathrm{Rub}_{\mathcal{L}_A}]^{vir} \cdot \eta^u)$?

$$\begin{tikzcd} \mathbb{P}\mathrm{Rub}_{\mathcal{L}_A} \arrow[d, "p"] \\ \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

One more ingredient: Chiodo classes

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\pi"'] & \mathcal{L} \arrow[l] \\ \overline{\mathcal{M}}^{r,K}_{g,A} \arrow[d, "\varepsilon"'] & \\ \overline{\mathcal{M}}_{g,n} & \end{tikzcd}$$

universal r-th root line bundle

$\overline{\mathcal{M}}^{r,K}_{g,A} = \{(C, p_1, \dots, p_n, \mathcal{L}) : \mathcal{L}^{\otimes r} \cong \omega_C^{\otimes K}(-\sum (a_i - K) p_i)\}$

$\overline{\mathcal{M}}_{g,n} \rightsquigarrow \boxed{\mathrm{Ch}^{K,r,d}_{g,A} := r^{2d-2g+1} \cdot \varepsilon_* c_d(-R^*\pi_* \mathcal{L}) \in CH^d(\overline{\mathcal{M}}_{g,n})}$

⇝ explicit formula in tautological ring [JPPZ], following Computations of Chiodo

⇝ polynomial in r for $r \gg 0$.

Conjecture (CGHMS)

For all $g, K, u \geq 0$, $A \in \mathbb{Z}^n$ as above:

$$p_*\left([\mathbb{P}\mathrm{Rub}_{\mathcal{L}_A}]^{\mathrm{vir}} \cdot \eta^u\right) = \underset{\uparrow \text{ take coeff. of } r^u}{[r^u]} \cdot \mathrm{Ch}_{g,A}^{K,r,g+u} \in CH^{g+u}(\overline{\mathcal{M}}_{g,n})$$

Evidence

- $u=0$: $\quad DR_g(A) = [r^0]\, \mathrm{Ch}_{g,A}^{K,r,g}$ known [JPPZ]
- Computer checks for $g=0$, $K>0$:

LHS $\mathbb{P}\mathrm{Rub}_{\mathcal{L}_A} \overset{g=0}{=} \mathbb{P}\mathcal{MS}_{g,\mu}$ $\left(\begin{smallmatrix}\text{GRC}\\ \text{satisfied}\end{smallmatrix}\right)$

$\rightsquigarrow$ cycle implemented in [diffstrata] by replacing $\eta \rightsquigarrow \psi$-classes + boundary

RHS Formula for Chiodo class in [admcycles]

- Proof for $K=0$!

Thm (GGHMS)

The conjecture is true for $K=0$

Pf [Fan–Wu–You] compute LHS in terms of cycles

$$\mathrm{Ch}_{g,A(r)}^{K,r,\bullet} = \sum_{d\geq 0} \mathrm{Ch}_{g,A(r)}^{K,r,d},$$

where

$$A(r)_i = \begin{cases} a_i & \text{, for } a_i \geq 0 \\ r+a_i & \text{, for } a_i < 0 \end{cases}$$

$\uparrow$ "high age insertions"

We obtain:

$$p_*\left([\overline{\mathcal{M}}_{g,A}(\mathbb{P}^1, 0, \infty)^\sim]^{\mathrm{vir}} \cdot \psi_\infty^u\right)$$

$$\underset{[\mathrm{FWY}]}{=} [r^u]\left[\prod_{i: a_i<0} \frac{1}{1-\frac{a_i}{r}\psi_i}\, \mathrm{Ch}_{g,A(r)}^{0,r,\bullet}\right]_{\mathrm{codim}\, g+u}$$

$$\underset{[\mathrm{GLN}]}{=} [r^u]\left[\mathrm{Ch}_{g,A}^{0,r,\bullet}\right]_{\mathrm{codim}\, g+u} = [r^u]\, \mathrm{Ch}_{g,A}^{0,r,g+u}$$

where we used a property of the Chiodo class proven in [Giacchetto–Lewański–Norbury]. □