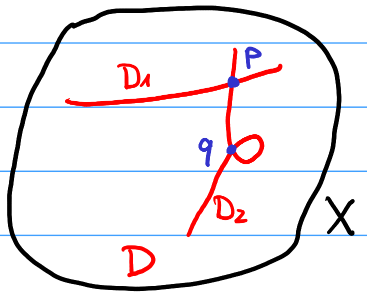
Logarithmic tautological rings (j/w. R. Pandharipande, D. Ranganathan, P. Spelier)

§0 Motivation

(X, D) smooth space with normal crossings divisor (smooth log smooth)



E.g. $(X = \overline{\mathcal{M}}_{g,n}\ ,\ D = \partial\overline{\mathcal{M}}_{g,n})$

mod. space of stable curves

locus of singular curves

$\rightsquigarrow$ stratification $X = \bigsqcup_\sigma S_\sigma$ into loc. closed $S_\sigma \subseteq X$.

$\boxed{Q1}$ How to intersect classes $[\overline{S_\sigma}] \in CH^*(X)$?

$\rightsquigarrow$ nice combinatorial formalism ?

E.g. $[D_1]\cdot[D_2] = [p] \in CH^2(X)$ above.

Next problem sometimes $U = X \setminus D$ is canonical, but it's (partial) compactification X is not!

Exa $U = \mathcal{A}_g$: moduli space of princ. polarized abel. var of dim g

$\rightsquigarrow X = \overline{\mathcal{A}}_g$: different birat'l models $(\overline{\mathcal{A}}_g^{PC}, \overline{\mathcal{A}}_g^{Alexeev}, \ldots)$

In good cases:

$$\begin{tikzcd} & \exists\, \widehat{X} \arrow[dl, dashed] \arrow[dr, dashed] & \\ X_1 & & X_2 \\ & U \arrow[ul, phantom, "\supseteq"] \arrow[ur, phantom, "\subseteq"] & \end{tikzcd}$$

any two X_1, X_2 receive map from common space $\widehat{X}$ and $\widehat{X} \to X_i$ is a log blow-up

think: iterated blow-up of smooth strata closures

Def (Holmes–Pixton–S.)

(X, D) smooth nc pair.

$$\rightsquigarrow \log CH^*(X, D) := \varinjlim_{\substack{(\widehat{X}, \widehat{D}) \to (X, D) \\ \text{log blow-up}}} CH^*(\widehat{X})$$

trans. maps = pullb. $\widehat{\pi}^*$

$\widehat{\widehat{X}} \xrightarrow{\widehat{\pi}} \widehat{X} \longrightarrow X$

logarithmic Chow ring

Basic properties

- $\log CH^*(X,D) = \{(\hat{X},\alpha) : \hat{X} \to X \text{ log blow-up}, \alpha \in CH^*(\hat{X})\} / (\hat{X},\alpha) \sim (\hat{\hat{X}}, \hat{\pi}^*\alpha)$ for $\hat{\hat{X}} \xrightarrow{\hat{\pi}} \hat{X} \to X$.

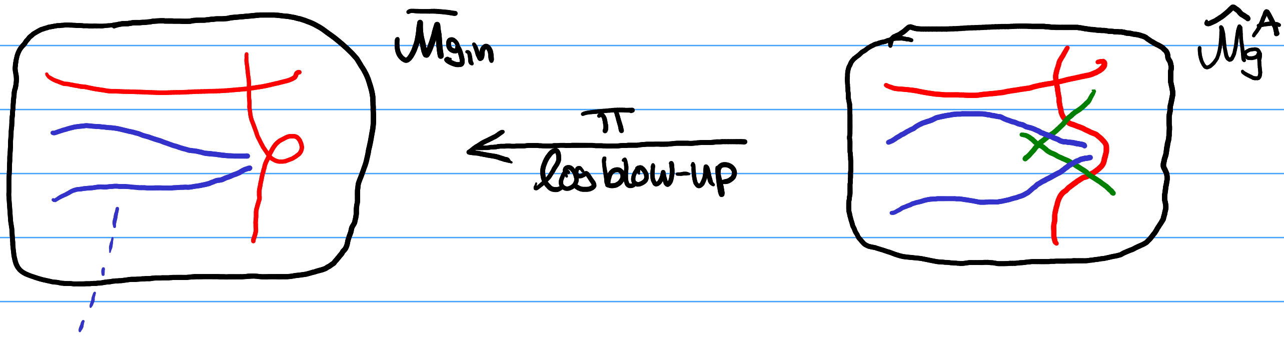
- $\log CH^*(X,D)$ is $\mathbb{Q}$-algebra,

$$\begin{aligned} CH^*(X) &\hookrightarrow \log CH^*(X,D) \\ \alpha &\longmapsto [(X,\alpha)] \end{aligned}$$

- $$\begin{aligned} \log CH^*(X,D) &\twoheadrightarrow CH^*(X) \qquad \mathbb{Q}\text{-linear} \\ [(\hat{X} \xrightarrow{\pi} X, \alpha)] &\longmapsto \pi_*\alpha. \end{aligned}$$

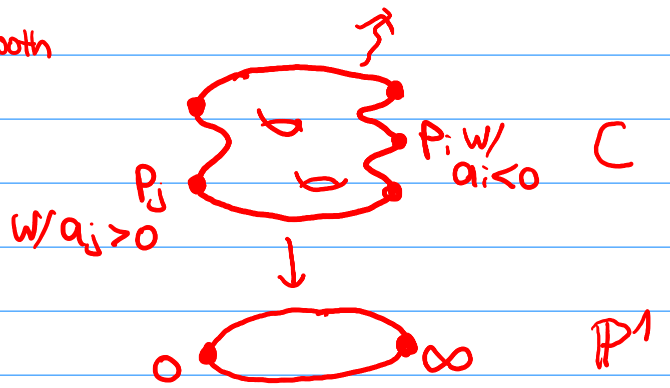
Applications & History

- [HPS] defined the logarithmic double ramification cycle



$DR_g^\circ(A) = \{(C, p_1, \dots, p_n) : \mathcal{O}_C(\sum a_i p_i) \cong \mathcal{O}_C\}$ $\qquad \widehat{DR}_g(A) \in CH^g(\widehat{\mathcal{M}}_g^A)$

$A = (a_1, \dots, a_n) \in \mathbb{Z}^n$ with $\sum a_i = 0$ $\qquad$ smooth $\qquad$ [Holmes]



$$\rightsquigarrow \log DR_g(A) = [(\widehat{\mathcal{M}}_g^A, \widehat{DR}_g(A))] \in \log CH^g(\overline{\mathcal{M}}_{g,n})$$

$$\downarrow \pi_*$$

$$DR_g(A) \in CH^g(\overline{\mathcal{M}}_{g,n}).$$

• [HPS]

$$\log CH^{2g}(\overline{\mathcal{M}}_{g,n}) \ni$$

$$\log DR_g(A) \cdot \log DR_g(B) = \log DR_g(A) \cdot \log DR_g(A+B)$$

$\boxed{\text{False for } DR_g}$

<u>Idea</u> $\begin{matrix} \mathcal{O}_C(\sum a_i p_i) \cong \mathcal{O}_C \\ \mathcal{O}_C(\sum b_i p_i) \cong \mathcal{O}_C \end{matrix} \iff \begin{matrix} \mathcal{O}_C(\sum a_i p_i) \cong \mathcal{O}_C \\ \mathcal{O}_C(\sum (a_i+b_i) p_i) \cong \mathcal{O}_C \end{matrix}$

• [Molcho – Pandharipande – S.]

$\boxed{\text{False for } \mathrm{div}CH^*}$

$$DR_g(A) \in \mathrm{div}\log CH^*(\overline{\mathcal{M}}_{g,n})$$

sub-$\mathbb{Q}$-algebra of $\log CH^*$ gen. by $\log CH^1$.

<u>Conjecture</u> [MPS] $\log DR_g(A) \in \mathrm{div}\log CH^*(\overline{\mathcal{M}}_{g,n})$

↳ proven by [Molcho-Ranganathan, Holmes-Schwarz]

• [Cavalieri – Markwig – Ranganathan]

$\boxed{\text{unknown for } DR_g}$

$$\text{Double Hurwitz Number}_g(A) = \int_{\widehat{\mathcal{M}}_g^A} \log DR_g(A) \cdot br_{g,A}.$$

• [Holmes – Molcho – Pandharipande – Pixton – S.]
calculate $\log DR_g(A)$ in terms of log-tautological classes on $\overline{\mathcal{M}}_{g,n}$.

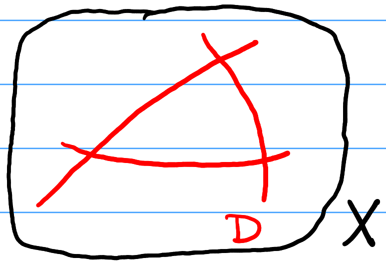
$\boxed{Q2}$ (PRSS, in preparation)

What are log-tautological classes ?

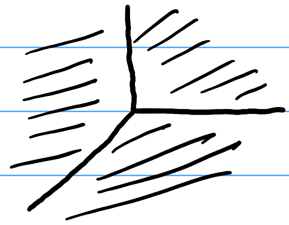
§1 Cone stacks & Artin fans

$\mathbb{G}_m \simeq \mathbb{C}^*$

Case 1 X smooth toric variety with torus $T \cong \mathbb{G}_m^n \overset{\text{open}}{\subseteq} X$, $D = X \setminus T$

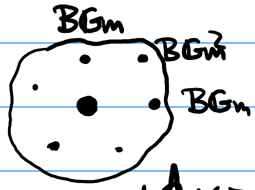


$\rightsquigarrow$



Σ_X fan

$\downarrow q_X$



$\mathcal{A}_X = [X/T]$

Thm (Brion) $\exists$ isom.

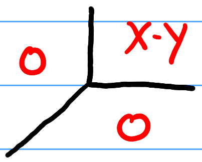
$$\begin{tikzcd} SPP^*(\Sigma_X) \arrow[d, "\Phi", "\cong"'] \arrow[dr] & \\ CH^*(\mathcal{A}_X) \arrow[r, "q_X^*"] & CH^*(X) \end{tikzcd}$$

Strict piecew. polynomials on Σ_X

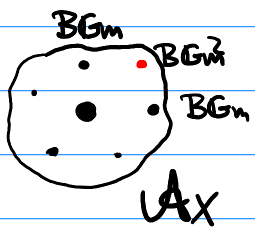
→ surjective with kernel $= \langle L(\Sigma) \rangle$

linear fct. on Σ

Exa



$= f \implies \Phi(f) = [B\mathbb{G}_m^2]$



Moreover

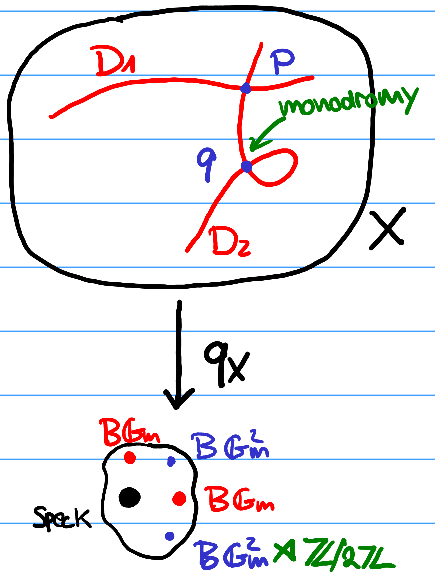
log blow-ups of X (or $\mathcal{A}_X$)

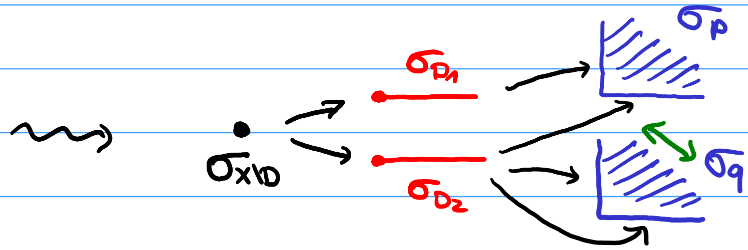
$\widehat{=}$ subdivisions of Σ_X

Case 2 (X, D) smooth nc pair

Idea
- étale locally around $P \in (X,D)$: D looks toric $\rightsquigarrow \Sigma_P$
- étale patches glue to (X,D) $\rightsquigarrow$ Σ_P glue to Σ_X

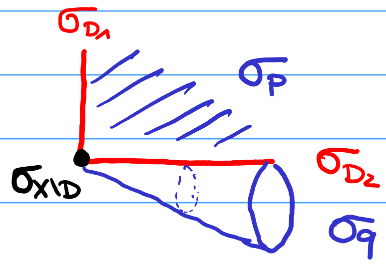
$$(X,D) \downarrow (\mathcal{A}_X, \mathcal{D})$$





$\Sigma_{(X,D)}$: cone stack

[Cavalieri–Chan–Ulirsch–Wise]



$\mathcal{A}_{(X,D)}$ Artin fan

[Abramovich–Chen–Marcus –Ulirsch–Wise]

Moreover

Subdivisions $\hat{\Sigma} \to \Sigma_{(X,D)}$ $\hat{=}$ log blow-ups

$$\begin{tikzcd} (\hat{X}, \hat{D}) \arrow[r] \arrow[d, "q_{\hat{X}}"] \arrow[dr, phantom, "\square"] & (X,D) \arrow[d, "q_X"] \\ (\hat{\mathcal{A}}, \hat{\mathcal{D}}) \arrow[r] & (\mathcal{A}_X, \mathcal{D}) \end{tikzcd}$$

Summary $\quad (X,D) \rightsquigarrow \Sigma_{(X,D)}$ cone stack

$$\begin{array}{c} (X,D) \\ \downarrow q_X \\ (\mathcal{A}_X,\mathcal{D}) \end{array} \quad \text{Artin fan}$$

Thm [MPS]

$\exists$ isom.

$$SPP^*(\Sigma_{(X,D)}) \xrightarrow[\sim]{\Phi} CH^*(\mathcal{A}_X) \xrightarrow{q_X^*} CH^*(X)$$

$$PP^*(\Sigma_{(X,D)}) \xrightarrow[\sim]{\Phi^{\log}} \log CH^*(\mathcal{A}_X,\mathcal{D}) \xrightarrow{q_X^*} \log CH^*(X,D)$$

both q_X^* no longer surjective

functions

$\Sigma_{(X,D)} \to \mathbb{R}$ compatible with

face maps & polynomial

(on all cones / on some subdivision $\hat{\Sigma} \to \Sigma_{(X,D)}$)

SPP* $\qquad$ PP*

Image of $q_X^* \circ \Phi^{(\log)}$: normally decorated strata classes

classes of strata closures in X (or $\hat{X}$)

decorated by Chern classes of

normal bundles.

$\rightsquigarrow$ $\boxed{Q1}$

§2 Applications to moduli spaces of curves

Def The <u>small log-tautological ring</u> $\log R^*_{sm}(\overline{\mathcal{M}}_{g,n})$

is the $\mathbb{Q}$-subalgebra of $\log CH^*(\overline{\mathcal{M}}_{g,n})$ gen. by

- $R^*(\overline{\mathcal{M}}_{g,n}) \subseteq CH^*(\overline{\mathcal{M}}_{g,n}) \subseteq \log CH^*(\overline{\mathcal{M}}_{g,n})$ taut. classes

- $\mathrm{im}\left(\Phi^{\log} : PP^*(\mathcal{M}_{g,n}^{trop}) \longrightarrow \log CH^*(\overline{\mathcal{M}}_{g,n})\right)$

$= \Sigma_{(\overline{\mathcal{M}}_{g,n},\partial\overline{\mathcal{M}}_{g,n})}$ $\quad$ moduli space of tropical curves

Thm (HMPPS)

$$\log DR_g(A) = \Big[\exp\big(\eta + \Phi^{\log}(f_L)\big)\cdot \Phi^{\log}(f_P)\Big]_g \in \log R^g_{\mathrm{Sm}}(\overline{\mathcal{M}}_{g,n})$$

$\eta = \sum \frac{a_i^2}{2}\psi_i \in R^1(\overline{\mathcal{M}}_{g,n})$ $\quad$ $f_L, f_P \in PP^*(\mathcal{M}_{g,n}^{trop})$ $\quad$ codim g part.

Thm (PRSS)

$$\Phi^{\log} : PP^*(\mathcal{M}_{0,n}^{trop}) \twoheadrightarrow \log CH^*(\overline{\mathcal{M}}_{0,n})$$

is surjective, kernel = gen. by $WDVV_{0,n}^{PP}$

piecew. linear fcts. on $\mathcal{M}_{0,n}^{trop}$ mapping to WDVV-rel's as norm. decorated strata classes.

Idea of proof

Construction of [Kapranov] $\rightsquigarrow$ $\overline{\mathcal{M}}_{0,n}$ = Chow quot. of $G(2,n)$ by $\mathbb{G}_m^n/\mathbb{G}_m$

$\Rightarrow$ $\exists$ smooth q.proj. toric variety $X_{0,n}$ with torus T: $\begin{pmatrix} p_1 & p_2 & \cdots & p_n \\ q_1 & q_2 & \cdots & q_n\end{pmatrix} \curvearrowleft GL_2$

$$\overline{\mathcal{M}}_{0,n} \xhookrightarrow{i} X_{0,n} \longrightarrow [X_{0,n}/T] = \mathcal{A}_{X_{0,n}}$$

$\uparrow/(\mathbb{G}_m^n/\mathbb{G}_m)$ $\quad$ $G(2,n) \xhookrightarrow{\text{Plücker}} \mathbb{P}^{\binom{n}{2}-1}$ $\quad$ $\cong \mathcal{A}_{\overline{\mathcal{M}}_{0,n}}$ $\rightsquigarrow$ Artin fans of $\overline{\mathcal{M}}_{0,n}$ & $X_{0,n}$ coincide

$\Rightarrow \{\log \text{ blow-ups } \widehat{\mathcal{M}} \to \overline{\mathcal{M}}_{0,n}\} \,\hat{=}\, \{\text{subdiv of } \mathcal{M}_{0,n}^{trop} = \Sigma_{X_{0,n}}\} \,\hat{=}\, \{\log \text{ blow-ups } \widehat{X} \to X_{0,n}\}$

Check i^* induces isom. of CH^* on all strata $\xRightarrow[\text{exact sequence}]{\text{Fulton's blow-up}}$ remains true for $\hat{i} : \widehat{\mathcal{M}} \to \widehat{X}$.

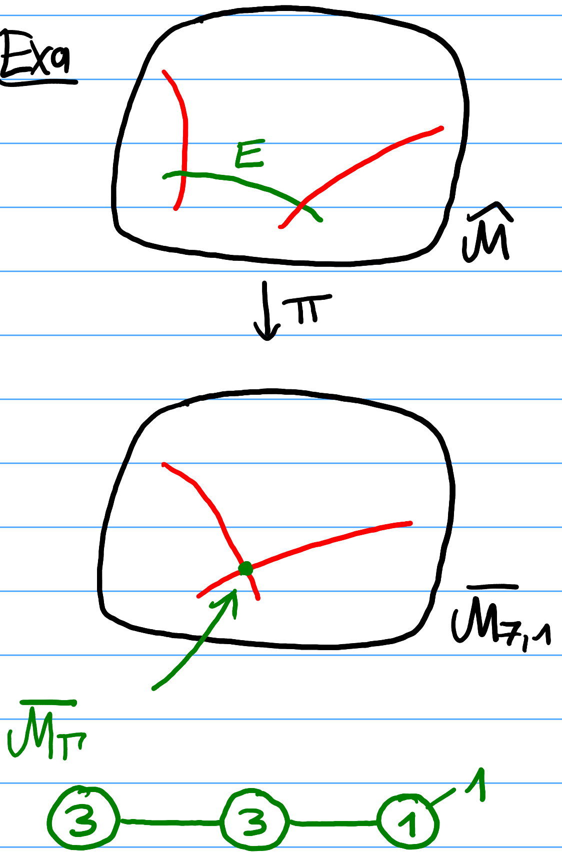
$$\Rightarrow \log CH^*(\overline{\mathcal{M}}_{0,n}) = \log CH^*(X_{0,n}) \overset{\text{Brion}}{=} PP^*(\underbrace{\Sigma_{X_{0,n}}}_{=\mathcal{M}_{0,n}^{trop}}) \Big/ \big(\underbrace{L(\Sigma_{0,n})}_{\overset{\text{Check}}{=} WDVV_{0,n}^{PP}}\big) \quad \square$$

§3 Larger log-tautological rings

Problem

Some natural classes are missing from $\log R^*_{sm}(\overline{\mathcal{M}}_{g,n})$

Exa



$\rightsquigarrow$
$$\begin{tikzcd} E \arrow[r, hook, "i"] \arrow[d, "\pi_E"'] & \widehat{\mathcal{M}} \\ \overline{\mathcal{M}}_\Gamma \end{tikzcd}$$
$\alpha = \kappa_1 \otimes 1 \otimes 1$

$\rightsquigarrow [\widehat{\mathcal{M}}, i_* \pi_E^* \alpha] \in \log CH^2(\overline{\mathcal{M}}_{7,1})$
looks tautological,
but not in $\log R^*_{sm}(\overline{\mathcal{M}}_{7,1})$.

Idea (X, D) snc (for simplicity)

Let $\widehat{X} \xrightarrow{\pi} X$ corr. to $\widehat{\Sigma} \to \Sigma_{(X,D)}$
$\hat\sigma \mapsto \sigma$ ($\hat\sigma \in \widehat{\Sigma}$, $\sigma \in \Sigma_{(X,D)}$)

$$\begin{tikzcd} \overline{S}_{\hat\sigma} \arrow[r, hook, "i_{\hat\sigma}"] \arrow[d, "p_{\hat\sigma}"] & \widehat{X} \arrow[d, "\pi"] \\ \overline{S}_\sigma \arrow[r, hook] & X \end{tikzcd}$$

strata closures

$$\begin{tikzcd} & \overline{S}_{\hat\sigma} \arrow[dl, "p_{\hat\sigma}"'] \arrow[r, hook, "i_{\hat\sigma}"] \arrow[d, "q_{\hat\sigma}"] & \widehat{X} \arrow[d, "q_{\widehat{X}}"] \\ \overline{S}_\sigma & B_{\hat\sigma} \arrow[r, hook, "\text{closed}"] & A_{\widehat{X}} \end{tikzcd}$$

Thm (PRSS)

$\exists$ isom.

$$\Psi : \{ f \in sPP^*(\widehat{\Sigma}) : f \equiv 0 \text{ outside } \mathrm{Star}_{\hat\sigma} \}$$
$$\downarrow \cong$$
$$CH_*(B_{\hat\sigma})$$

Def $\log R^*(X,D)$ is $\mathbb{Q}$-vector space gen. by

$$[\hat\sigma, f, \alpha] := [(\hat X, (\iota_{\hat\sigma})_*(p_{\hat\sigma}^*\alpha \cap q_{\hat\sigma}^* \Psi(f))] \in \log CH^*(X,D)$$

$\hat\sigma$ cone in some $\hat\Sigma \to \Sigma$

f as above

$\alpha \in CH^*(\overline{S}_\sigma)$ decoration $\boxed{\text{CHOICE}}$

Then

- allowing $\alpha =$ poly in κ, ψ-classes in $CH^*(\overline{\mathcal{M}}_\Gamma)$ $\rightsquigarrow$ class from above is in $\log R^*(\overline{\mathcal{M}}_{g,n})$
- $\log R^*(X,D)$ completely determined by $(CH^*(\overline{S}_\sigma))_{\sigma \in \Sigma(X,D)}$ & maps between them.
- $\alpha \in CH^*(\overline{S}_\sigma)$ allowed to be arb. elem. of $CH^*(\overline{S}_\sigma)$

  $\rightsquigarrow \log R^*_{CH}(X,D) = \log CH^*(X,D)$

  $\rightsquigarrow [\hat\sigma, f, \alpha]$ give additive generating set.

Thank you for your attention!