# Logarithmic intersection theory
## From toric varieties to moduli of curves

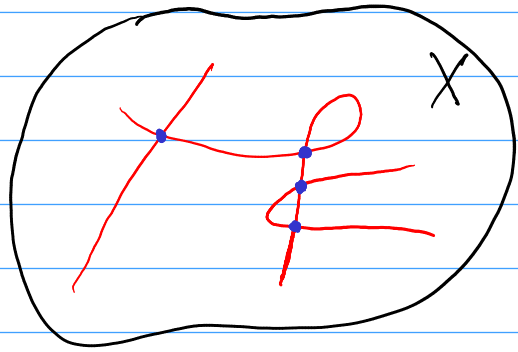
## §0. Motivation

$X$ smooth variety (or Deligne-Mumford stack)

$D \subseteq X$ normal crossing (nc) divisor $\rightsquigarrow$ étale locally $D$ is a union of coord. hyperplanes

$D = \{x_1 \cdot x_2 \cdots x_r = 0\} \subseteq \mathbb{A}^n$

branches of $D$




$X^{=2}$

$X^{=1}$

**Goal** study stratification of $D$ and intersections of strata classes

**Examples**

- $X$ = toric variety with torus $T = (\mathbb{C}^*)^n \subseteq X$, $D = X \setminus T$
- $X = \overline{\mathcal{M}}_{g,n}$ moduli space of stable curves
  $D = \partial \overline{\mathcal{M}}_{g,n}$ boundary (i.e. locus of singular curves)
- more generally: $X$ = moduli of admissible covers, multi-scale differentials, ...
- Bott-Samuelson varieties (resolving Schubert cells in Grassmannians)

**Stratification of $X$**

**Def** For $x \in X$ let $\mathrm{rank}(x) = \#$ branches of $D$ intersect. at $x$

A stratum $S$ of $X$ of codim $r$ is a connected component of $X^{=r} = \{x \in X : \mathrm{rank}(x) = r\}$.

**Prop** Strata $S \subseteq X$ locally closed, and $X = \bigsqcup S$.

**Exa**
- $X$ toric $\rightsquigarrow$ $S$ = $T$-orbits of pts on $X$
- $X = \overline{\mathcal{M}}_{g,n}$ $\rightsquigarrow$ $S$ = locus of curves with given shape

# Intersection theory of strata classes

Reference: [Fulton]

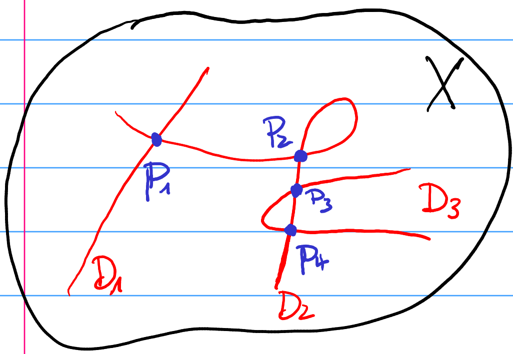
$$CH^*(X) = \bigoplus_{\substack{V \subseteq X \\ \text{subvariety}}} \mathbb{Q}\cdot[V] \Big/ \sim_{\text{rat'l equivalence}}$$ Chow ring

graded $\mathbb{Q}$-algebra w.r.t. intersection product

deg $[V]$ = codim$(V)$

e.g. $[V]\cdot[W] = \sum [Z_i]$ if $V, W$ intersect transv. $V \cap W = \bigsqcup Z_i$

## Example




$[D_1]\cdot[D_2] = [P_1]$

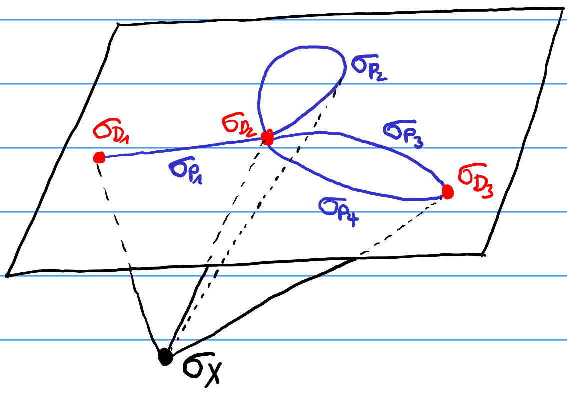
$[D_1]\cdot[D_3] = 0$

$[D_2]\cdot[D_3] = [P_3] + [P_4]$

less clear: $[D_i]^2 = ??$

**Question 1** How to encode the combinatorics of $(X, D)$ to allow intersection calculations of strata?

**Answer** cone stack $\Sigma_{(X,D)}$ (= "tropicalization of $(X,D)$")




Strata of $(X,D)$

$\updownarrow$ inclusion-reversing

Cones of $\Sigma_{(X,D)}$

This cone stack also contains information on $X \setminus D$:

**Thm** [CGP, Theorem 5.8]

$X$ smooth and proper DM stack of dim $d$, $D \subseteq X$ nc divisor

$$\Rightarrow \quad \mathrm{Gr}^W_{2d} \underbrace{H^{2d-k}(X \setminus D; \mathbb{Q})}_{\text{singular cohomology}} \cong \underbrace{H_{k-1}(\Sigma_{(X,D)}; \mathbb{Q})}_{\text{cohomology of bdry complex}}.$$

graded piece wrt. mixed Hodge structure

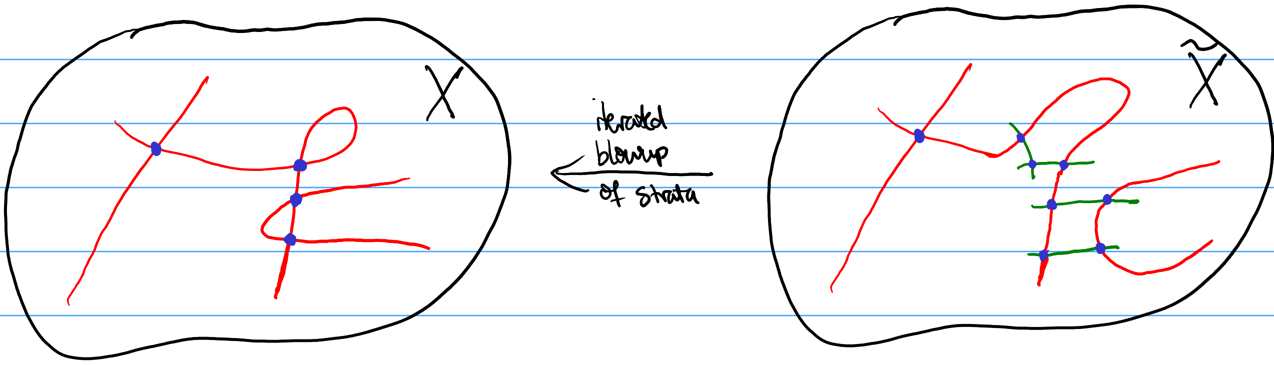
# Iterated blowups

**Sometimes:** $U = X \setminus D$ canonical, but some choice for $X$
(e.g. $X$ = any compactific. of $U$ w/ normal cross. bdy)

$\rightsquigarrow$ modifications of $X$ not changing $U$?

**Def** A boundary blowup $(\tilde{X}, \tilde{D}) \xrightarrow{\pi} (X, D)$ is given by
$\pi : \tilde{X} = Bl_{\bar{S}} X \longrightarrow X$ for a non-singular stratum closure $\bar{S} \subseteq X$
$\tilde{D} = \pi^{-1}(D)$ total transform of $D$

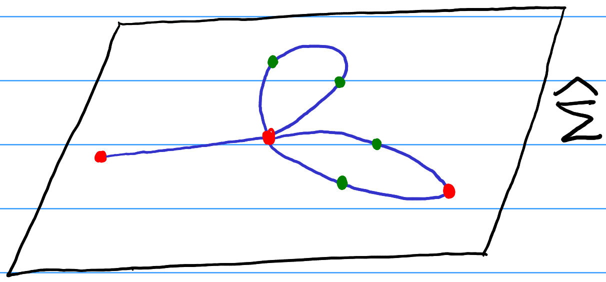
**Prop** $(\tilde{X}, \tilde{D})$ is again an nc pair, & isomorphism over $U = X \setminus D$.




**Application** can make $D$ into simple normal crossing divisor (snc)
all comp. of $D$ are smooth

**Question 2** How to encode such a choice of blowup?

**Answer** subdivision $\hat{\Sigma} \to \Sigma_{(X,D)}$



# Logarithmic intersection theory

**Def** (Log Chow ring, [HPS])

$$\log CH^*(X,D) = \varinjlim_{\substack{(\tilde X,\tilde D)\to(X,D)\\ \text{iterated bd. blowup}}} CH^*(\tilde X)$$

$$\begin{tikzcd} \pi : \tilde X \arrow[r] & X \end{tikzcd}$$
$$\Downarrow$$
$$\begin{tikzcd} \pi^* : CH^*(X) \arrow[r, shift left] & CH^*(\tilde X) \arrow[l, shift left, "\pi_*"] \end{tikzcd}$$

**Facts** • Colimit is filtered:

$$\begin{tikzcd} & \exists\, \tilde{\tilde X} \arrow[dl, dashed, "f"'] \arrow[dr, dashed, "g"] & \\ \tilde X \arrow[dr] & & \tilde X' \arrow[dl] \\ & X & \end{tikzcd}$$

$$\Rightarrow \log CH^*(X,D) = \bigcup_{(\tilde X,\tilde D)\to(X,D)} CH^*(\tilde X) \Big/ [\tilde X,\alpha]\sim[\tilde X',\beta] \text{ iff } \exists\, \tilde{\tilde X} : f^*\alpha = g^*\beta.$$

• $CH^*(X) \hookrightarrow \log CH^*(X)$ with inverse given by pushforward $\pi_*$

$$\log CH^*(X) \longrightarrow CH^*(X)$$
$$[\tilde X \xrightarrow{\pi} X, \alpha] \longmapsto \pi_*\alpha$$

$\rightsquigarrow$ Why care about $\log CH^*$?

**Example** (Logarithmic double ramification cycle, [H])

Given $A=(a_1,\dots,a_n)\in\mathbb{Z}^n$ with $\sum a_i = 0$

$\rightsquigarrow$ $DRL(A) = \{(C,p_1,\dots,p_n)\in \mathcal{M}_{g,n} : \mathcal{O}(\sum a_i p_i)\cong\mathcal{O}\}$ → compactify in $\overline{\mathcal{M}}_{g,n}$?

$$\begin{tikzcd}
 & & & \mathfrak{Pic}^{[0]} = \{(C,p_1,\dots,p_n,\mathcal{L}) : \mathcal{L}/C \text{ line bdle deg}=0 \text{ on all comp. of } C\} \arrow[d] \\
S_A^\diamond \cap S_0 = DRL_A^\diamond \subseteq \mathcal{M}^\diamond_{g,A} \arrow[rrru, "S_A^\diamond"] \arrow[rrr, "\pi_0"] \arrow[d, "\iota"'] & & & \overline{\mathcal{M}}_{g,n} \arrow[u, bend left, "S_A'\ (\mathcal{L}=\mathcal{O}(\Sigma a_ip_i))"] \arrow[u, bend right, "S_0\ (\mathcal{L}=\mathcal{O})"'] \\
\widehat{\mathcal{M}} \arrow[rrru, "\pi"', "\text{iterated bdry blowup}"] & & &
\end{tikzcd}$$

proper

non-canonical

**Def** (Holmes)

$$\log DR_g(A) = [\widehat{\mathcal{M}}, \iota_*(S_A^\diamond)^*[S_0(\overline{\mathcal{M}}_{g,n})]] \in \log CH^*(\overline{\mathcal{M}}_{g,n})$$

$$DR_g(A) = \pi_* \log DR_g(A) \qquad \text{(usual) DR-cycle.}$$

## Advantages of log DR

- [HPS] $\log DR_g(A) \cdot \log DR_g(B) = \log DR_g(A) \cdot \log DR_g(A+B)$

[false for DR]

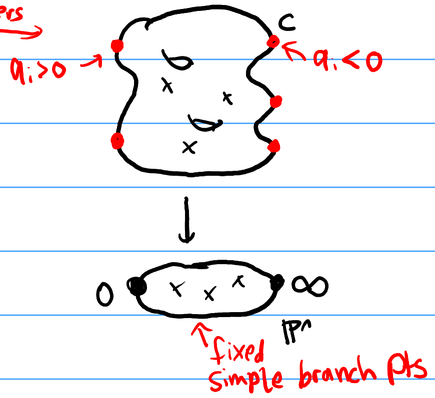
- [MPS, HS] $\log DR_g(A) \in \mathrm{div}\log CH^*(\overline{\mathcal{M}}_{g,n})$

sub-algebra of log $CH^*$ gen. by log $CH^1$

[$DR_g(A) \notin \mathrm{div}\, CH^*(\overline{\mathcal{M}}_{g,n})$ in general]

- [CMR] Double Hurwitz numbers $\xrightarrow{\text{counting covers of } \mathbb{P}^1}$

$$H_g(A) = \int_{\overline{\mathcal{M}}_{g,n}} \log DR_g(A) \cdot \underbrace{br_{2g-3+n}}_{\in \log CH^{2g-3+n}(\overline{\mathcal{M}}_{g,n})}$$




[No known formula for $H_g(A)$ using $DR_g(A)$]

**Question 3** How to encode classes in $\log CH^*(X, D)$ coming from strata closure in some blowup?

**Answer** piecewise polynomials on $\Sigma_{(X,D)}$

## Overview of the course

§0 Motivation
§1 Toric varieties
§2 Cone stacks & Artin fans
§3 Piecewise polynomials & associated (log) Chow classes
§4 Applications to moduli spaces of curves

# §1 Toric varieties

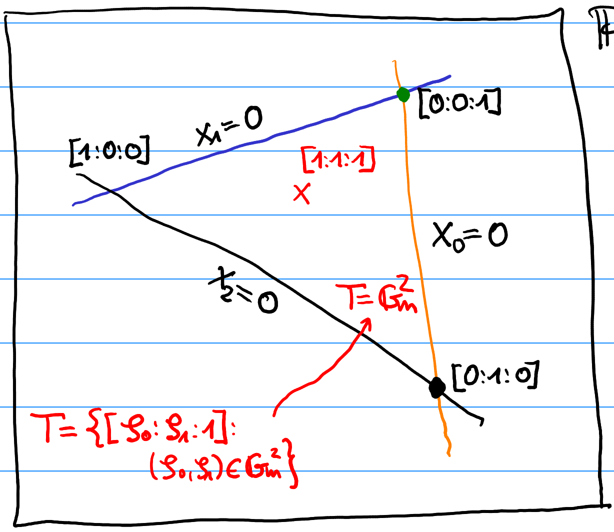
Reference: [CLS]

**Def** $X$ variety$/k$ of dim $n$ is toric if it contains an algebraic torus $T = \mathbb{G}_m^n \subseteq X$ as an open subset, such that $T \curvearrowright T$ extends to $T \curvearrowright X$.

**Exa** $\mathbb{A}^n$, $\mathbb{P}^n$, $\mathrm{Tot}(\mathcal{O}(m)) \longrightarrow \mathbb{P}^n$

## Orbits and cones

**Exa**




$\mathbb{P}^2 \ni [x_0:x_1:x_2]$

↳ $\mathbb{P}^2$ decomposes into $T$-orbits

↳ $\partial\mathbb{P}^2 = \mathbb{P}^2 \setminus T$ is snc divisor, strata of $(\mathbb{P}^2, \partial\mathbb{P}^2)$ $=$ $T$-orbits

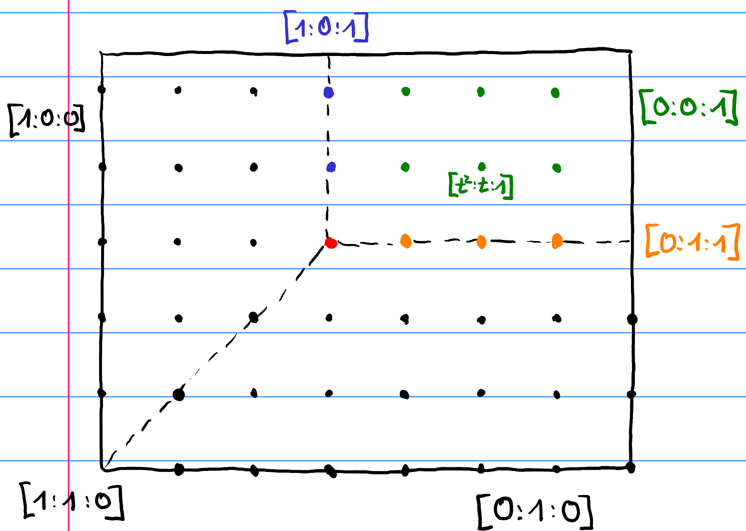
**Q** How to organize strata of $(X, \partial X)$ nicely?

**A** via one-parameter subgroups of $T$!

$$\lambda^{(a,b)} : \mathbb{G}_m \longrightarrow T, \quad t \mapsto [t^a : t^b : 1]$$

$$v = (a,b) \in \mathbb{Z}^2 = N = \mathrm{Hom}(T, \mathbb{G}_m)^\vee$$

**Q** Depending on $v \in \mathbb{Z}^2$, where is $\lim_{t\to 0} \lambda^v(t) \in \mathbb{P}^2$?




**A** Determined by fan $\Sigma_{\mathbb{P}^2}$ in $N_{\mathbb{R}}$

# Cones & fans

$M = \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}^n$ lattice of characters

$N = M^\vee$ lattice of 1-PSGs

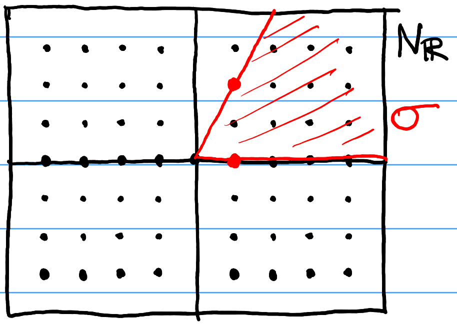
$M_\mathbb{R} \cong N_\mathbb{R}^\vee \cong \mathbb{R}^n$

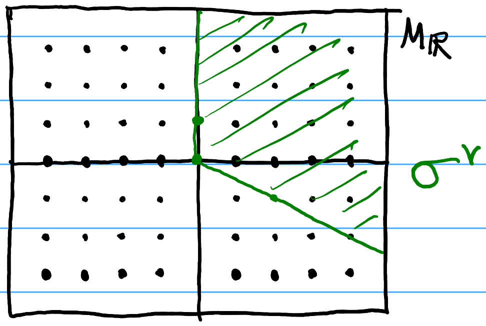
<u>Def</u> $\mathcal{V} \subset N$ finite $\rightsquigarrow$ $\sigma = \mathrm{Cone}(\mathcal{V}) = \{\sum_{v \in \mathcal{V}} a_v \cdot v : a_v \geq 0\}$ <u>rational polyhedral cone</u>

$\sigma$ <u>strictly convex</u> if $\sigma \cap (-\sigma) = \{0\}$

$\sigma^\vee = \{m \in M_\mathbb{R} : \langle u, m \rangle \geq 0 \quad \forall u \in \sigma\} \subseteq M_\mathbb{R}$ <u>dual cone</u>

<u>Exa</u>



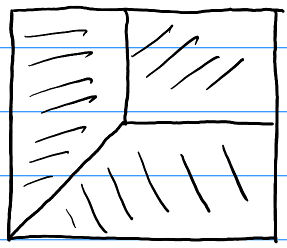

<u>Def</u> $\sigma' \leq \sigma$ <u>face</u> if $\exists\, m \in \sigma^\vee$ st. $\sigma' = \{u \in \sigma : \langle u, m \rangle = 0\}$

↪ write $\sigma' \prec \sigma$

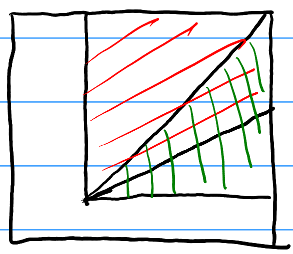
<u>Def</u> A finite collection $\Sigma$ of cones in $N_\mathbb{R}$ is called a <u>fan</u> if

- → all $\sigma \in \Sigma$ are rat'l polyhedral, strictly convex
- → for $\sigma' \prec \sigma$ face, $\sigma \in \Sigma \Rightarrow \sigma' \in \Sigma$
- → for $\sigma_1, \sigma_2 \in \Sigma \Rightarrow \sigma_1 \cap \sigma \prec \sigma_1, \sigma_2$

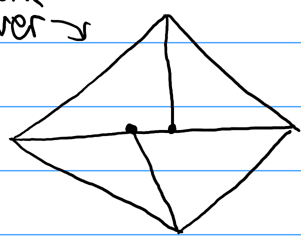
<u>Exa</u>



<u>Non-Exa</u>



Cone over ↘



**Fact** Given a normal toric variety $X$ there exists a fan $\Sigma_X$ in $N_{\mathbb{R}}$ and bijection

$$\Psi : \Sigma_X \xrightarrow{\sim} \{T\text{-orbits } S \subseteq X\}$$ orbit-cone correspondence

determined by property that for $\sigma \in \Sigma$, $u \in \underbrace{\mathrm{relint}(\sigma)}_{= \sigma \setminus \bigcup_{\sigma' \precneq \sigma \text{ proper face}} \sigma'} \cap N$:

$$\Psi(\sigma) = T.\left(\lim_{t\to 0} \lambda^u(t)\right) =: S_\sigma$$

**Properties**

- $\sigma' \prec \sigma \in \Sigma_X \iff S_\sigma \subseteq \overline{S_{\sigma'}}$ (inclusion-reversing)

  $\rightsquigarrow \overline{S_{\sigma_1}} \cap \overline{S_{\sigma_2}} = \begin{cases} \overline{S_{\sigma_1+\sigma_2}} & \text{if } \sigma_1+\sigma_2 \in \Sigma \\ \varnothing & \text{otherwise} \end{cases}$ $\boxed{\text{Question 1 for toric var.}}$

  intersections of strata closures determined by $\Sigma$

- $\dim_{\mathbb{R}} S_\sigma = n - \dim_{\mathbb{R}} \sigma$, in particular:

  $S = T \longleftrightarrow \sigma = \{0\} \in \Sigma$

  $\{\text{divisorial strata } S_\tau\} \longleftrightarrow \tau \in \Sigma(1)$ rays

  $\{T\text{-fixed points}\} \longleftrightarrow \sigma \in \Sigma(n)$ maximal cones

- Properties of $X$ can be checked on $\Sigma_X$:

  ↳ $X$ proper $\iff \underbrace{|\Sigma_X|}_{\text{support}} = \bigcup_{\sigma\in\Sigma_X} \sigma = N_{\mathbb{R}}$ $\rightsquigarrow$ $\Sigma_X$ complete

  ↳ $X$ finite quot. singularities $\iff$ $\Sigma$ simplicial : $\forall \sigma \in \Sigma : \#\sigma(1) = \dim \sigma$

  $\iff$ $\sigma$ spanned by $\dim \sigma$ vectors in $N$.

  ↳ $X$ smooth $\iff$ $\Sigma$ smooth : $\forall \sigma \in \Sigma : \sigma \cap N \cong \mathbb{N}^{\dim \sigma}$

  $\iff$ $\sigma$ spanned by $\dim \sigma$ vectors in $N$ that extend to basis of $N$

- $X$ smooth $\Rightarrow \partial X = X \setminus T$ is snc divisor, strata of $(X, \partial X) = T$-orb. $S_\sigma$

  $x \in S_\sigma \rightsquigarrow \mathrm{rank}(x) = \dim \sigma$, branches of $\partial X$ at $x \mathrel{\widehat{=}}$ rays of $\sigma$.

# Toric intersection theory

Reference: [Brion]

<u>Fact</u> $X$ smooth toric variety

$\Rightarrow CH^*(X) = \langle [\bar{S}_\sigma] : \sigma \in \Sigma_X \rangle$ generated by strata classes

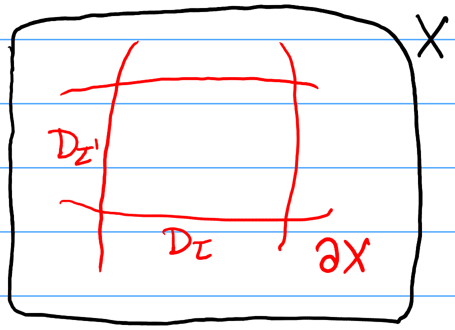
<u>Pf</u> Excision sequence ([Fulton]), all $S_\sigma \cong \mathbb{G}_m^{n-\dim\sigma}$. $\square$

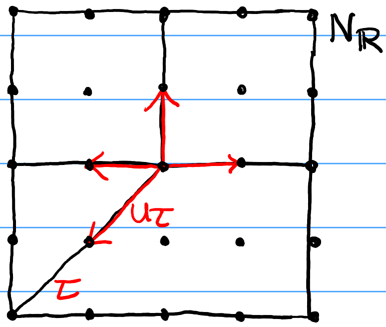
<u>Q</u> What are the relations & ring structure?

codim 1: $CH^1(X) = \bigoplus_{\substack{D \subseteq X \\ \text{divisor}}} \mathbb{Q}\cdot[D] \Big/ \langle \operatorname{div}(f) : X \overset{f}{\underset{\text{rat'l function}}{\dashrightarrow}} \mathbb{A}^1 \rangle$

$$\begin{array}{ccccc} m \in M = \operatorname{Hom}(T, \mathbb{G}_m) \rightsquigarrow \chi^m : & T & \longrightarrow & \mathbb{G}_m & \text{rat'l funct.} \\ & \cap\!\shortmid & & \cap\!\shortmid & \\ & X & \dashrightarrow & \mathbb{A}^1 & \end{array}$$

$\operatorname{div}(\chi^m)$ supported on $\partial X = \bigcup_{\tau \in \Sigma(1)} D_\tau$



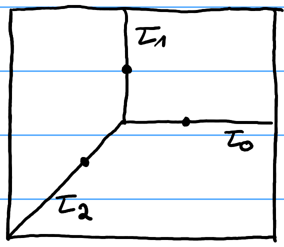

$\tau \in \Sigma(1)$ ray

$u_\tau \in \tau$ primitive generator of $\tau \cap N$

$\overset{\text{check}}{\rightsquigarrow}$ $\operatorname{div}(\chi^m) = \sum_{\tau \in \Sigma(1)} \langle u_\tau, m \rangle \cdot [D_\tau]$

value of $m$ at prim. gen. of $\tau$

<u>Exa</u> $X = \mathbb{P}^2$




$m = (1,0) \rightsquigarrow 1\cdot[D_{\tau_0}] + (-1)\cdot[D_{\tau_2}] = 0$

$m = (0,1) \rightsquigarrow 1\cdot[D_{\tau_1}] + (-1)\cdot[D_{\tau_2}] = 0$

$\rightsquigarrow [D_{\tau_2}] = [D_{\tau_1}] = [D_{\tau_0}] = H \in CH^1(\mathbb{P}^2) = \mathbb{Q}\cdot H.$

<u>Fact</u> These span all relations in codim. 1

$$CH^1(X) = \bigoplus_{\tau \in \Sigma(1)} \mathbb{Q}\cdot[D_\tau] \Big/ \langle \operatorname{div}(\chi^m) : m \in M \rangle$$

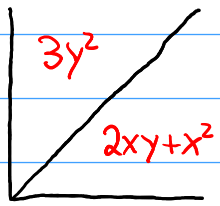
**Slogan** relations of toric bdry. div $D_\tau$
$\hat{=}$ linear functions $\chi \in L(\Sigma) := M$

**To generalize to $CH^*$** interpret $[D_\tau]$ as function on $\Sigma$ as well!

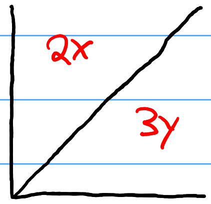
**Def** Given a fan $\Sigma$, a *strict piecewise polynomial* (spp) on $\Sigma$ is a continuous function $\varphi : |\Sigma| \longrightarrow \mathbb{R}$ which restricts to a polynomial w/ rat'l coefficients (↑ in coordinate functions on $N_\mathbb{R}$) on each cone $\sigma \in \Sigma$

$\rightsquigarrow$ $sPP^*(\Sigma)$ = ring of spp's on $\Sigma$
$sPL(\Sigma) = sPP^1(L)$ strict piecewise linear (spl) fcts. on $\Sigma$

**Exa**




**Non-Exa**



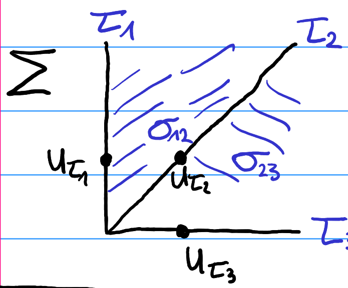

**Def** Let $\Sigma$ simplicial, $\tau \in \Sigma(1)$ w/ prim. gen. $u_\tau \in \tau \cap N \cong \mathbb{N}$.
$\rightsquigarrow$ $\exists!\ \varphi_\tau \in SPL(\Sigma)$ with

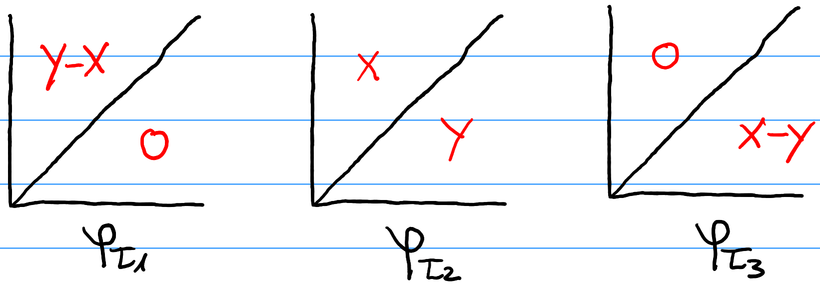
$$\varphi_\tau(u_{\tau'}) = \begin{cases} 1, & \tau' = \tau \\ 0, & \tau' \in \Sigma(1) \setminus \{\tau\} \end{cases}$$

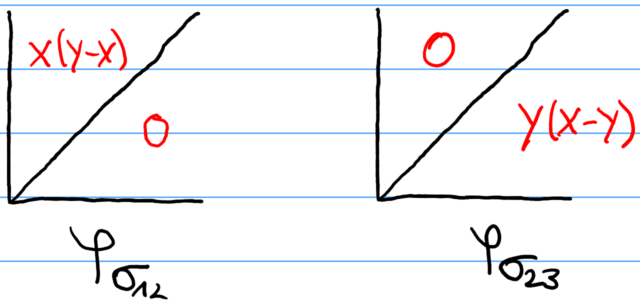
More generally: for $\sigma \in \Sigma$:

$$\varphi_\sigma = \prod_{\tau \in \sigma(1)} \varphi_\tau$$

**Exa**




$$\boxed{\begin{array}{l}\text{In general:}\\ \varphi_\sigma \text{ supported on}\\ \mathrm{Star}(\sigma) = \{\sigma' \in \Sigma : \sigma \prec \sigma'\}\end{array}}$$

**Prop** For $\Sigma$ simplicial:
$$\mathbb{Q}^{\Sigma(1)} \xrightarrow{\sim} sPL(\Sigma)$$
$$(a_\tau)_{\tau\in\Sigma(1)} \longmapsto \sum_\tau a_\tau \cdot \varphi_\tau$$

In particular:
$$CH^1(X) = sPL(\Sigma_X)/L(\Sigma_X)$$

**Thm** [Brion, Payne] $X$ smooth toric variety, then
$$\begin{array}{ccc} CH^*(X) & = & sPP^*(\Sigma_X)/(L(\Sigma)) \\ \rotatebox{90}{$\in$} & & \\ [\overline{S}_\sigma] & \longmapsto & [\varphi_\sigma] \end{array}$$

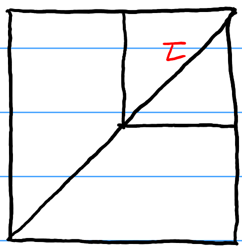
$\boxed{\text{Question 3.1 for toric var.}}$

Equivariant variant:

torus-equivariant Chow ring [EG]
$$\begin{array}{ccc} CH^*_T(X) & = & sPP^*(\Sigma_X) \\ \parallel & & \\ CH^*(\underbrace{[X/T]}_{\text{Artin fan of } X \text{ (see §2)}}) & & \end{array}$$

See [Fantechi] for more on Artin stacks!

**Exercise** $X = Bl_{[0:0:1]}\mathbb{P}^2 \supseteq E = \overline{S}_\tau$ except. div.




**Show** $[E]^2 = -[pt]$.

# Stanley-Reisner presentation of $sPP^*(\Sigma)$

Let $\Sigma$ simplicial with rays $\varsigma \in \Sigma(1)$.

**Def** Inside $R_\Sigma = \mathbb{Q}[x_\varsigma : \varsigma \in \Sigma(1)]$ define the Stanley-Reisner ideal

$$\mathcal{I}_\Sigma = (x_{\varsigma_1} \cdot x_{\varsigma_2} \cdot \ldots \cdot x_{\varsigma_m} : \operatorname{cone}(\varsigma_1, \ldots, \varsigma_m) \notin \Sigma) \subseteq R_\Sigma$$

**Prop** The map $R_\Sigma \xrightarrow{\Psi} sPP^*(\Sigma)$, $x_\varsigma \mapsto \varphi_\varsigma$ is surjective, $\ker(\Psi) = \mathcal{I}_\Sigma$.

$\Rightarrow sPP^*(\Sigma) \cong R_\Sigma / \mathcal{I}_\Sigma$.

**Sketch of pf** $f \in sPP^*(\Sigma) \rightsquigarrow$ Idea Pick $f$ apart into monomials to get preim. under $\Psi$

**Start** $f(0) \in \mathbb{Q} \subseteq R_\Sigma \rightsquigarrow f - \Psi(f(0))$ restricts to zero on $\Sigma(0)$.

**Step** For $m = 1, \ldots, n$ assume $f|_{\Sigma(m-1)} = 0$ and consider cones $\sigma \in \Sigma(m)$

let $\{\varsigma_1, \ldots, \varsigma_m\} = \sigma(1)$ rays. $\rightsquigarrow x_{\varsigma_1}, \ldots, x_{\varsigma_m}$ coordinates on $\sigma$

$$\Rightarrow f|_\sigma = \sum_{\underline{e}} \underset{\substack{\uparrow \\ \mathbb{Q}}}{a_{\underline{e}}} \cdot x_{\varsigma_1}^{e_1} \cdots x_{\varsigma_m}^{e_m}, \quad \underline{e} \in \mathbb{Z}_{\geq 0}^m$$

$$\Rightarrow f - \Psi\Big(\sum_{\sigma \in \Sigma(m)} \sum_{\underline{e}} a_{\underline{e}} \cdot x_{\underline{\varsigma}}^{\underline{e}}\Big) \text{ restricts to zero on } \Sigma(m)$$

**Induction**: $f \in \operatorname{im}(\Psi)$.

Process above: section $\tilde{\Psi} : sPP(\Sigma) \to R_\Sigma$ of $\Psi$

**Check**

$$(\tilde{\Psi} \circ \Psi)(x_{\underline{\varsigma}}^{\underline{e}}) = \begin{cases} x_{\underline{\varsigma}}^{\underline{e}} & \text{if } x_{\underline{\varsigma}}^{\underline{e}} \notin \mathcal{I}_\Sigma \\ 0 & \text{otherwise} \end{cases} \qquad \square$$

**Cor** $sPP^*(\Sigma)$ generated by $sPL(\Sigma)$ as $\mathbb{Q}$-algebra.

Picking apart a piecewise polynomial function

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# Toric varieties vs. fans – reloaded

**Before**: $X$ toric $\rightsquigarrow \Sigma_X$ fan ; now: converse direction

<u>Idea</u> · construct $X$ by gluing affine toric varieties $U_\sigma$ $(\sigma \in \Sigma)$

· face inclusions $\sigma' \preceq \sigma$ in $\Sigma \rightsquigarrow$ gluing data $U_{\sigma'} \hookrightarrow U_\sigma$

Let $\Sigma$ be a fan in $N_{\mathbb{R}}$, $\sigma \in \Sigma$

$\rightsquigarrow P_\sigma = \sigma^\vee \cap M = \{m \in M : \langle u, m\rangle \geq 0 \ \forall u \in \sigma\}$ is a monoid

$\sigma$ strongly convex $\Rightarrow$ $P_\sigma^{gp} = M$

<u>Def</u> The affine toric variety $U_\sigma$ assoc. to $\sigma$ is given by

$$U_\sigma = \operatorname{Spec} \underline{\Bbbk[P_\sigma]}$$

$$\Bbbk[P_\sigma] \cong \bigoplus_{m \in \sigma^\vee \cap M} \Bbbk \cdot \chi^m \quad \text{with } \chi^m \cdot \chi^{m'} = \chi^{m+m'}$$

<u>Note</u> $P_\sigma \to M \rightsquigarrow \Bbbk[P_\sigma] \longrightarrow \Bbbk[M]$

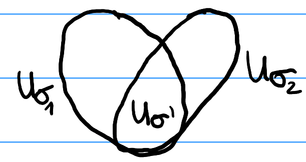
$$U_\sigma = \operatorname{Spec} \Bbbk[P_\sigma] \xhookleftarrow{\text{open}} \operatorname{Spec} \Bbbk[M] = T_M \quad \text{torus}$$

$$\operatorname{Spec} \Bbbk[M] \cong \operatorname{Spec} \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \cong \mathbb{G}_m^n$$

<u>Gluing construction</u>

$\sigma_1, \sigma_2 \in \Sigma \Rightarrow \sigma' = \sigma_1 \cap \sigma_2 \in \Sigma$ and

$\sigma' \preceq \sigma_i \rightsquigarrow \Bbbk[P_{\sigma'}] \hookleftarrow \Bbbk[P_{\sigma_i}] \rightsquigarrow U_{\sigma'} \xhookrightarrow{\text{open}} U_{\sigma_i}$




<u>Def</u> Toric variety $X = X_\Sigma$ associated to $\Sigma$ obtained by gluing $\{U_\sigma : \sigma \in \Sigma\}$ along $U_{\sigma'} \hookrightarrow U_\sigma$ from $\sigma' \preceq \sigma$

$$X_\Sigma = \varinjlim_{\sigma \in \Sigma} U_\sigma$$

<u>Example</u>

$\sigma = (\mathbb{R}_{\geq 0})^n$ positive orthant

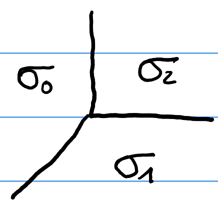
$\sigma^\vee = (\mathbb{R}_{\geq 0})^n \subseteq (\mathbb{R}^n)^\vee \cong \mathbb{R}^n$

$\sigma^\vee \cap M = \sigma^\vee \cap \mathbb{Z}^n = \mathbb{N}^n$

$\Bbbk[\sigma^\vee \cap M] = \Bbbk[x_1, \dots, x_n]$

$U_\sigma = \mathbb{A}^n$

<u>Exercise</u> Given fan $\Sigma$ below:




<u>Check</u> $U_{\sigma_i} \cong \mathbb{A}^2$, gluing gives usual charts:

$X_\Sigma = \mathbb{P}^2 = U_0 \cup U_1 \cup U_2$

**Theorem** There is an equivalence of categories

$$\left\{\begin{array}{c}\text{normal toric}\\ \text{varieties}\end{array}\right\} \xleftrightarrow{1:1} \{\text{fans } \Sigma\}$$

$$X \longmapsto \Sigma_X$$
$$X_\Sigma \longleftarrow\!\shortmid \Sigma$$

**Morphisms**

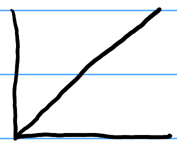
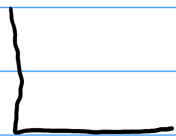
$X \supseteq T_X$ and $X' \supseteq T_{X'}$ toric varieties

$X \to X'$ toric if it induces group homomorph. $T_X \to T_{X'}$

$\Sigma$ in $N_\mathbb{R}$, $\Sigma'$ in $N'_\mathbb{R}$ fans

fan morphism $\Sigma \to \Sigma'$ is map $N \xrightarrow{\phi} N'$ of lattices s.t. $\forall \sigma \in \Sigma\ \exists \sigma' \in \Sigma' : \phi_\mathbb{R}(\sigma) \subseteq \sigma'$.

**Example**

$Bl_0 \mathbb{A}^2 \longrightarrow \mathbb{A}^2 \quad \longleftrightarrow$  $\xrightarrow{\phi = id_{N^2}}$ 

**More generally** $\Sigma, \hat{\Sigma}$ fans in $N_\mathbb{R}$

$\hat{\Sigma} \xrightarrow{id_N} \Sigma$ fan morphism

$\hat{\Sigma}$ refines $\Sigma$ if $|\hat{\Sigma}| = |\Sigma|$ and each $\hat{\sigma} \in \hat{\Sigma}$ is contained in some $\sigma \in \Sigma$

**Prop**

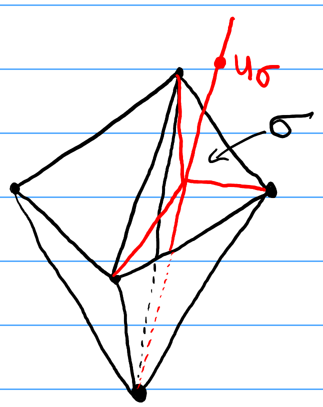
$$\left\{\begin{array}{c}\text{proper birational}\\ \text{toric morphisms}\\ X_{\hat{\Sigma}} \to X_\Sigma\end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{c}\text{refinements}\\ \hat{\Sigma} \text{ of } \Sigma\end{array}\right\}$$

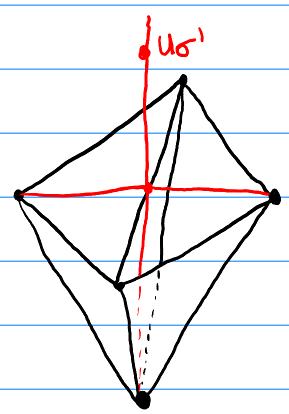
toric blow-ups

**Exa** $\Sigma$ smooth fan, $\sigma \in \Sigma$ with ray generators $u_\varsigma$

$\rightsquigarrow u_\sigma = \sum_{\varsigma \in \sigma(1)} u_\varsigma$ barycenter

$\rightsquigarrow$ star subdivision $\Sigma^*(\sigma) \to \Sigma \;\hat{=}\; Bl_{\overline{S_\sigma}} X_\Sigma \to X_\Sigma$




$\boxed{\text{Question 2 for toric var!}}$

# Logarithmic toric intersection theory

Recall:

$$\log CH^*(X,D) = \varinjlim_{\substack{\hat{X}\to X \\ \text{log blowup}}} CH^*(\hat{X})$$

What about $(X,D) = (X_\Sigma, \partial X_\Sigma)$ ?

$\rightsquigarrow \hat{X} = X_{\hat{\Sigma}}$ for $\hat{\Sigma} \to \Sigma$ refinement

$$CH^*(X_{\hat{\Sigma}}) = sPP^*(\hat{\Sigma}) / (L(\hat{\Sigma}))$$

strict piecewise polynomials

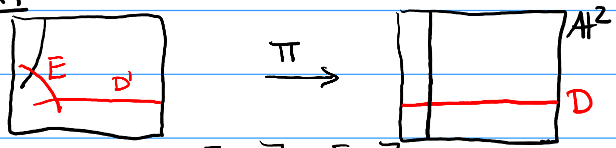
linear functions

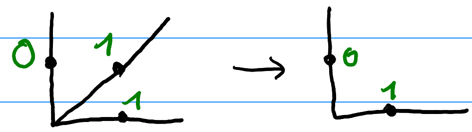
Q What about transition maps ?

Exa

Fan side

values of $\varphi$




$\rightsquigarrow \pi^*[D] = [D'] + [E]$

total transform　　strict transform

$\varphi(x,y) = x$ $\updownarrow$ $[D'] + [E]$　　　$\varphi(x,y) = x$ $\updownarrow$ $[D]$

Fact $X_{\hat{\hat{\Sigma}}} \xrightarrow{\pi} X_{\hat{\Sigma}}$ toric blow-up between smooth toric var.

$$\begin{array}{ccc} CH^*(X_{\hat{\Sigma}}) & = & sPP^*(\hat{\Sigma})/(L(\hat{\Sigma})) \\ \downarrow \pi^* & & \downarrow \varphi \mapsto \varphi \\ CH^*(X_{\hat{\hat{\Sigma}}}) & = & sPP^*(\hat{\hat{\Sigma}})/(L(\hat{\Sigma})) \end{array}$$

uses: $\varphi \in sPP^*(\hat{\Sigma})$
& $\hat{\hat{\Sigma}}$ refines $\hat{\Sigma}$
$\Rightarrow \varphi \in sPP^*(\hat{\hat{\Sigma}})$

Def $\Sigma$ fan

$$PP^*(\Sigma) = \varinjlim_{\substack{\hat{\Sigma}\to\Sigma \\ \text{refinement}}} sPP^*(\hat{\Sigma}) = \left\{ \varphi : |\Sigma| \to \mathbb{R} \;\middle|\; \begin{array}{l} \varphi \text{ continuous,} \\ \text{strict piecew. polynom. on} \\ \text{some subdiv. of } \Sigma \end{array} \right\}$$

Cor $\Sigma$ smooth fan

$$\Rightarrow \log CH^*(X_\Sigma) \cong PP^*(\Sigma)/(L(\Sigma))$$

$\boxed{\text{Question 3.2 for toric var.}}$

# §2. Cone stacks & Artin fans

**Input** $(X,D)$ w/ $X$ smooth, $D\subseteq X$ nc divisor

**Want**

Cone stack $\Sigma_{(X,D)}$ describing stratification

convex geometry

$\cong$

Artin fan $\mathcal{A}_{(X,D)}$

$\mathcal{A}_X$ = algebraic stack

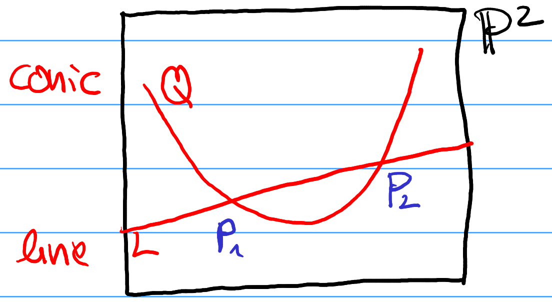
Intersect. thy of $(\hat{X},\hat{D})\longrightarrow(X,D)$ $\log CH^*(X,D)$

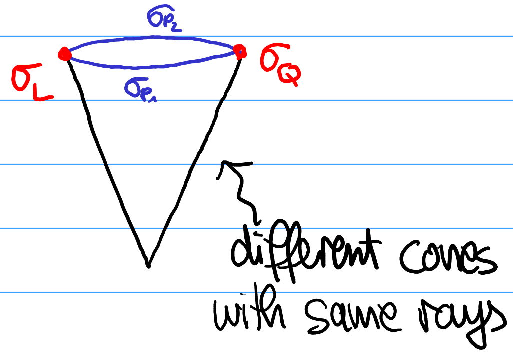
algebraic geometry

## Cone stacks

**Q** What problems arise when leaving toric world?

**Problem 1** Intersections w/ multiple components

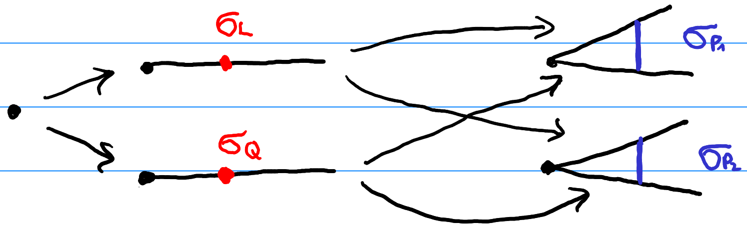


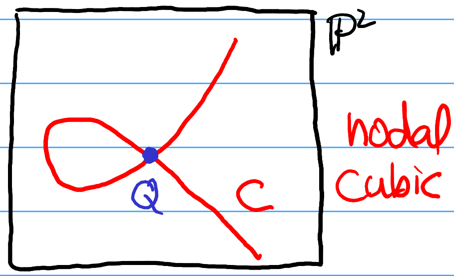

**Solution**

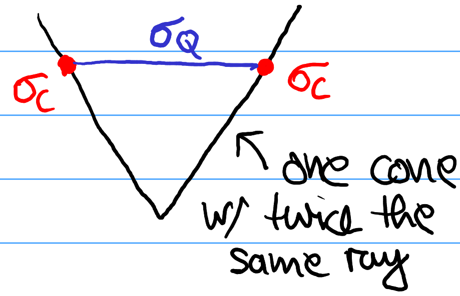
Abandon ambient space $N_{\mathbb{R}}$ of $\Sigma$

$\rightsquigarrow$ $\Sigma$ = collection of cones & face inclusions

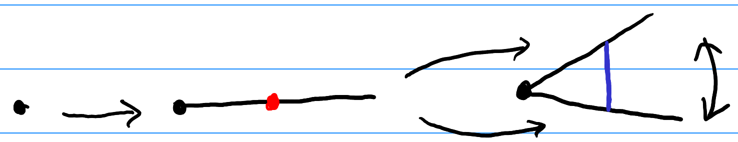


**Problem** self-intersections & monodromy




**Solution** diagram of cones can have multiple morphisms between objects (and, in particular, non-triv. automorph.)



**Def** [CCUW]

A cone stack is a category fibered in groupoids

$\Sigma$ category $\quad \sigma : \Sigma \longrightarrow RPC^f$ $\quad$ rat'l polyhedral cones w/ morphisms = face inclusions

$\alpha \longmapsto \sigma_\alpha$

$(\alpha \to \beta) \longmapsto (\sigma_\alpha \xrightarrow{\text{face}} \sigma_\beta)$

In other words, $\sigma$ satisfies:

→ $\tau \prec \sigma_\alpha$ face $\Rightarrow \exists \beta \to \alpha$ st. $\sigma_\beta \to \sigma_\alpha$ has image $\tau$

→ $\begin{matrix}\gamma \searrow \\ \beta \nearrow\end{matrix} \alpha$ st. $\sigma_\gamma \leq \sigma_\beta \Rightarrow \exists!$
$$\begin{tikzcd} \gamma \arrow[d, dashed] \arrow[dr] & \\ \beta \arrow[r] & \alpha \end{tikzcd} \quad \overset{\sigma}{\longmapsto} \quad \begin{tikzcd} \sigma_\gamma \arrow[d] \arrow[dr] & \\ \sigma_\beta \arrow[r] & \sigma_\alpha \end{tikzcd}$$

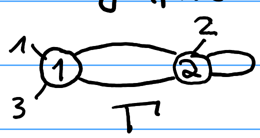
**Exa 0** Pictures above for $(\mathbb{P}^2, D \cup Q)$ and $(\mathbb{P}^2, C)$

**Exa 1** $\Sigma$ fan $\rightsquigarrow Cat(\Sigma)$ : $Ob = \{\sigma \in \Sigma\}$ $\rightsquigarrow$ $\underbrace{Cat(\Sigma) \longrightarrow RPC^f}_{= \text{ cone stack of } (X_\Sigma, \partial X_\Sigma)}$

$Mor = \{\sigma' \prec \sigma\}$

**Exa 2** (Moduli space of tropical curves [CCUW])

$\Sigma = \{\text{stable graphs } \Gamma\} \longrightarrow RPC^f$



$\longmapsto \sigma_\Gamma = \mathbb{R}_{\geq 0}^{E(\Gamma)} = \{\ell : E(\Gamma) \longrightarrow \mathbb{R}_{\geq 0}\}$ length assignments

**Exercise** (for experts)

What about morphisms $\Gamma' \to \Gamma \triangleq \sigma_{\Gamma'} \to \sigma_\Gamma$ ? $\rightsquigarrow$ Solution next time.

# Constructing the cone stack $\Sigma_{(X,D)}$

References [ACMUW, U]

For $x \in X$ $\overset{D \text{ nc divisor}}{\rightsquigarrow}$ $\begin{cases} \exists \text{ small étale neighbourhood } U_x \xrightarrow{u_x} X \\ \exists \text{ cone } \sigma_x \text{ w/ associated affine toric var. } V_{\sigma_x}: \end{cases}$

$$(x, U_x, u_x^{-1}(D)) \longrightarrow (y_{\sigma_x}, V_{\sigma_x}, \partial V_{\sigma_x})$$

étale map of pointed spaces respecting nc divisors

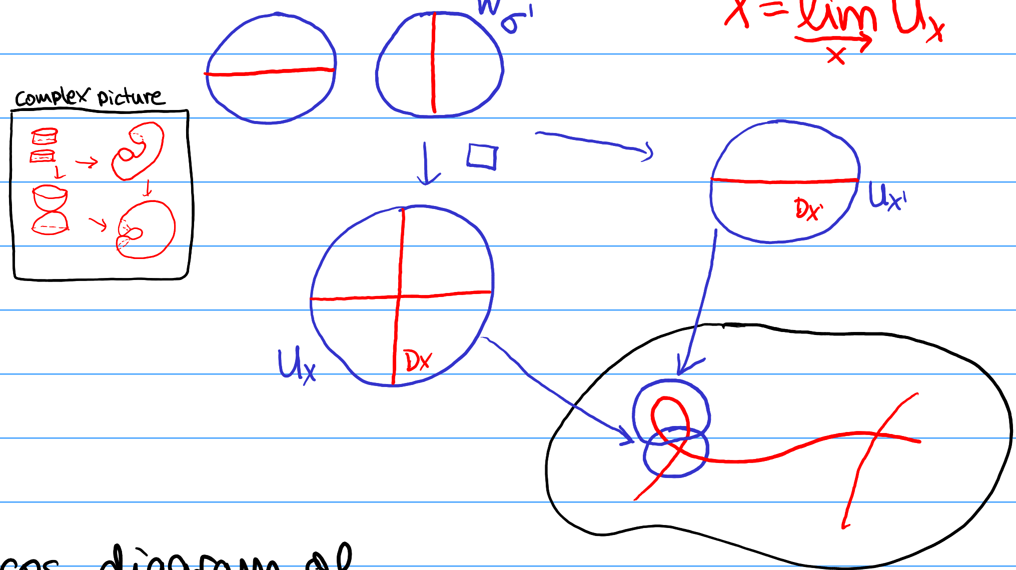
↑ pt in $S_{\sigma_x} \subseteq V_{\sigma_x}$ min. stratum.

Let $\mathcal{U} = \coprod_x U_x \longrightarrow X$ be resulting cover, $\mathcal{W} \longrightarrow \mathcal{U} \times_X \mathcal{U}$ further étale cover as above

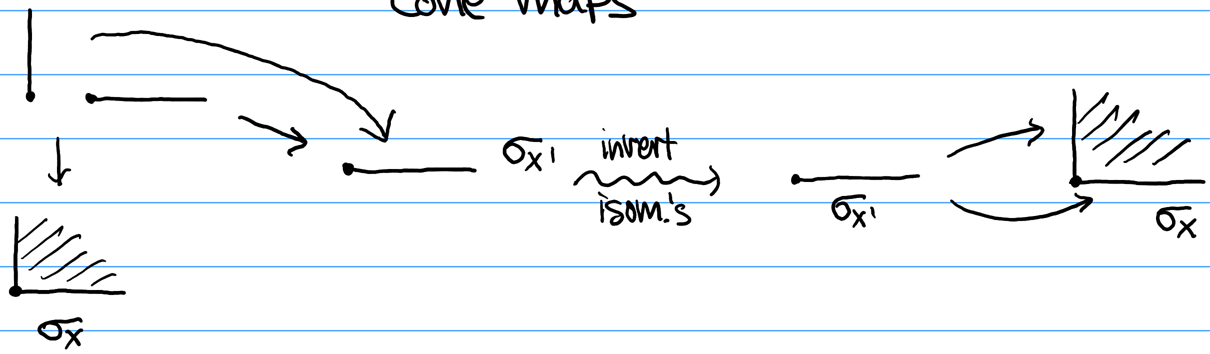
⇝ Have groupoid presentation $\mathcal{W} \rightrightarrows \mathcal{U} \longrightarrow X$ (fancy for: obtain $X$ by gluing schemes $U_x$ along $\mathcal{W}$)

⇝ $X = \varinjlim_x U_x$

Exa




⇝ induces diagram of cone maps




Def [ACMUW, Sect. 5.3]

The cone stack $\Sigma_{(X,D)}$ is defined from the presentation

$$\coprod_{\sigma'} \sigma' \rightrightarrows \coprod_x \sigma_x \longrightarrow \Sigma_{(X,D)} = \varinjlim_{x \in X} \sigma_x .$$

↑ informal

$\boxed{\text{Question 1 for } (X,D)}$

# Artin fans

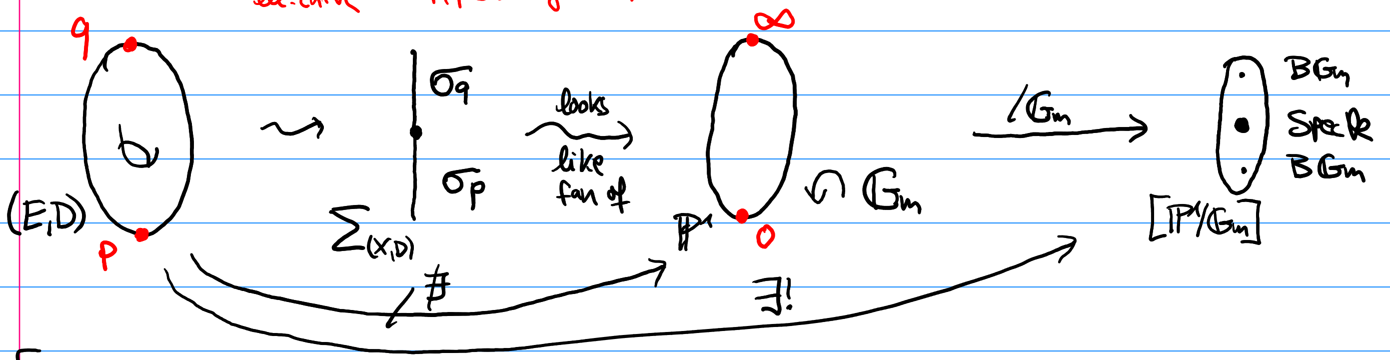
$\Sigma_{(X,D)}$ = Convex geometry + brain-melting category nonsense

Q How to get back to (algebraic) geometry?

A More brain-melting category nonsense: Artin stacks!

Exa $(X,D) = (E, p+q)$

ell. curve $\quad$ $p,q \in E$ (general)




## Exercise

The quotient stack $\mathcal{A}_n = [\mathbb{A}^n/\mathbb{G}_m^n]$ has universal property:

$$\{\text{maps } S \to \mathcal{A}_n\} \cong \{(\mathcal{L}_1, s_1, \dots, \mathcal{L}_n, s_n) : \mathcal{L}_i/S \text{ line bdle}, s \in H^0(S, \mathcal{L}_i)\}$$

$$E = (E \setminus \{q\}) \cup (E \setminus \{p\})$$

$(\mathcal{L} = \mathcal{O}(p), s_p) \downarrow \qquad \downarrow (\mathcal{L} = \mathcal{O}(q), s_q)$ $\quad$ $\operatorname{div}(s_q) = [q]$

$$[\mathbb{P}^1/\mathbb{G}_m] = [\mathbb{A}^1/\mathbb{G}_m] \cup [\mathbb{A}^1/\mathbb{G}_m]$$

## Def

$$(X,D) = \varinjlim_x U_x \longrightarrow \mathcal{A}_{(X,D)} = \varinjlim_x \underbrace{[V_{\sigma_x}/T_{\sigma_x}]}_{\text{Artin cone } \mathcal{A}_{\sigma_x}}$$

$V_{\sigma_x}$ $\quad$ $T_{\sigma_x} \subseteq V_{\sigma_x}$ torus $\quad$ Artin fan

Rmk 1) Map $X \longrightarrow \mathcal{A}_{(X,D)}$ is smooth, $D$ = preimage of $\mathcal{D} \subseteq \mathcal{A}_{(X,D)}$

2) Cone stacks & Artin fans are equivalent:

log stacks w/ surjective strict étale cover from disj. union of Artin cones

Thm [CCUW]

The functor

$$\text{Cone stacks} \longrightarrow \text{Artin fans}$$

$$(\{\sigma\}, \{\tau \to \sigma\}) \longmapsto \varinjlim_\sigma \mathcal{A}_\sigma$$

$\mathcal{A}_\sigma = [\operatorname{Spec} k[P_\sigma] / \operatorname{Spec} k[P_\sigma^{gp}]]$

is an equivalence of 2-categories.

Idea $\qquad \tau \longrightarrow \sigma \qquad$ face map of rat. polyh. cones

$\Leftrightarrow$ $\qquad \operatorname{Spec} k[P_\tau] \longrightarrow \operatorname{Spec} k[P_\sigma] \qquad$ open $T_\sigma$-equivariant embedding
Fan vs. Toric

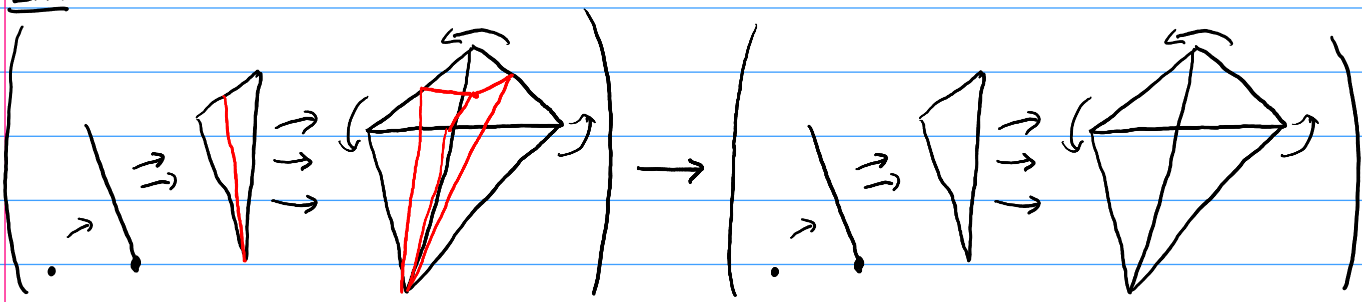
$\Leftrightarrow$ $\qquad \mathcal{A}_\tau \longrightarrow \mathcal{A}_\sigma \qquad$ strict open embedding
log geometry

□

## Subdivisions & log blow-ups

**Def** For cone stack $\Sigma$, a subdivision $\hat{\Sigma} \to \Sigma$ is choice of a subdivis. of each $\sigma \in \Sigma$, compatible w/ face maps (& automorph.) in $\Sigma$.

**Exa**



Then $\hat{\Sigma}$ is itself a cone stack, and thus $(\hat{\Sigma} \to \Sigma) \rightsquigarrow (\mathcal{A}_{\hat{\Sigma}} \longrightarrow \mathcal{A}_X)$.
log blowup.

**Exa** $\Sigma = \Sigma_{(X,D)} \rightsquigarrow$ subdivision $\hat{\Sigma} \longrightarrow \Sigma_{(X,D)}$ gives

$$\begin{tikzcd} \hat{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{A}_{\hat{\Sigma}} \arrow[d] \\ X \arrow[r] & \mathcal{A}_X \end{tikzcd} \qquad (*)$$

**Def** A log blow-up $(\hat{X}, \hat{D}) \longrightarrow (X, D)$ is a map obtained as a fiber product $(*)$ from some subdivision $\hat{\Sigma} \to \Sigma_{(X,D)}$.

$\boxed{\text{Question 2 for } (X,D)}$

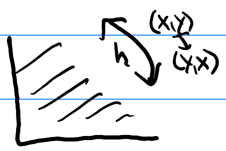
↳ see meme on next page

## §3. Piecewise polynomials & log Chow classes

$\Sigma$ cone stack. $\qquad f \in PP(\Sigma) \qquad$ Piecewise

**Def** A [strict] piecewice polynomial $f \in sPP(\Sigma)$ is a polynomial fct. $f_\sigma : \sigma \to \mathbb{R}$ for each $\sigma \in \Sigma$ s.t. $\forall \sigma' \xrightarrow{h} \sigma$ in $\Sigma$: $f_\sigma \circ h = f_{\sigma'}$.

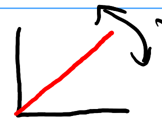
**Exa**

$\Sigma$ 

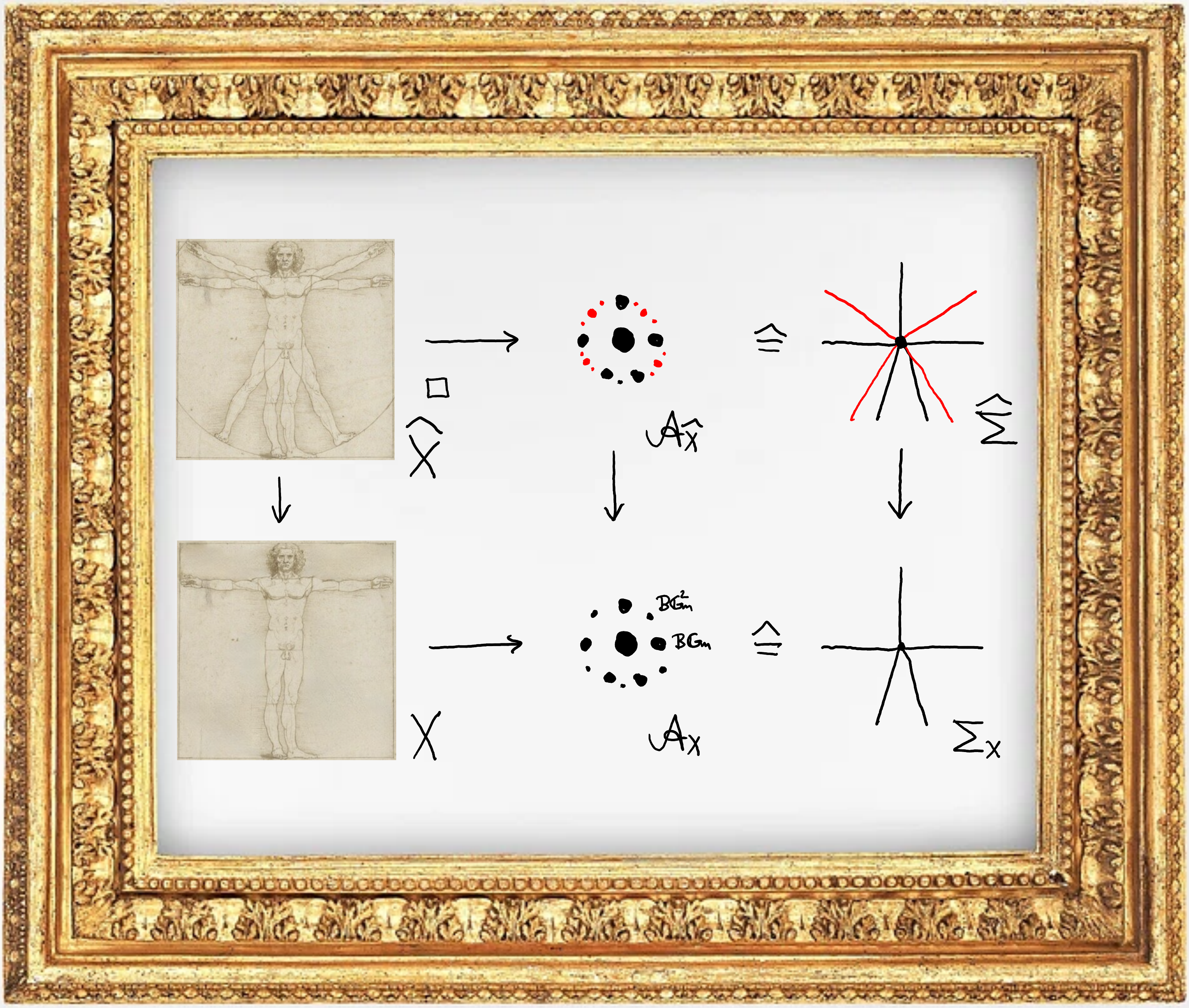
$f = x+y,\ x-y \in sPP^*(\Sigma)$

$g = x \notin sPP^*(\Sigma) \rightsquigarrow g \circ h = y \neq g$

$\hat{f} = \min(x,y) \in PP^*(\Sigma)$

$\in sPP^*($  $)$

# The vitruvian fan



# (Logarithmic) Chow classes from piecewise polynomials

$\Sigma$ smooth cone stack, $\sigma \in \Sigma \rightsquigarrow \mathcal{S}_\sigma \subseteq \mathcal{A}_\Sigma$ associated stratum

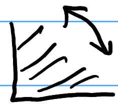
$\tau \in \Sigma(1) \rightsquigarrow \mathcal{S}_\tau$ codim 1 $\rightsquigarrow \mathcal{D}_\tau = \overline{\mathcal{S}_\tau} \subseteq \mathcal{A}_X$ divisor

Thus, like for toric varieties, we have map

$$\Phi_1 : sPP^1(\Sigma) \longrightarrow CH^*(\mathcal{A}_X), \quad f \mapsto \sum_{\tau \in \Sigma(1)} f(u_\tau) \cdot [\mathcal{D}_\tau]$$

Chow groups of Artin stacks, see [K]

For $X$ toric:
$$\begin{tikzcd} \mathrm{Sym}^* sPP^1(\Sigma_X) \arrow[r, two heads] \arrow[dr, "\mathrm{Sym}\,\Phi_1"'] & sPP^*(\Sigma_X) \arrow[d, dashed, "\exists!\,\Phi \text{ ring map}"] \\ & CH^*(\mathcal{A}_X) \end{tikzcd}$$
surjective (**)

No longer true for general cone stacks: $\Sigma =$ 

$sPP^1(\Sigma) = \mathbb{Q} \cdot (x+y)$, but $sPP^2(\Sigma) = \mathbb{Q} \cdot (x^2+y^2) \oplus \mathbb{Q} \cdot (xy)$.

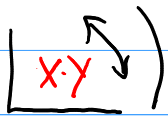
**Def** [HS] Choose refinement $\hat{\Sigma} \xrightarrow{\pi} \Sigma$ s.t. $\mathrm{Sym}^* sPP^1(\hat{\Sigma}) \twoheadrightarrow sPP^*(\hat{\Sigma})$.

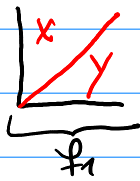
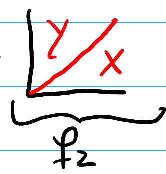
$\rightsquigarrow \mathcal{A}_{\hat{\Sigma}} \xrightarrow{\pi} \mathcal{A}_\Sigma$ proper

(**)$\rightsquigarrow$

$$\begin{tikzcd} sPP^*(\hat{\Sigma}) \arrow[r, "\Phi_{\hat{\Sigma}}"] & CH^*(\mathcal{A}_{\hat{\Sigma}}) \arrow[d, "\pi_*"] \\ sPP^*(\Sigma) \arrow[u, "\pi_\Sigma^*"] \arrow[r, dashed, "=:\Phi"] & CH^*(\mathcal{A}_\Sigma) \end{tikzcd}$$

**Check**
$\Phi$ well-defined ring morphism!

**Exa** $\Phi\left( \right.$   $\left. \right) = ?$

$\rightsquigarrow$ $sPP^1(\hat{\Sigma}) = \mathbb{Q} \cdot \underbrace{\text{}}_{f_1}$   $\oplus\ \mathbb{Q} \cdot \underbrace{\text{}}_{f_2}$  

$\hat{\Sigma} \longrightarrow \Sigma$ $\quad \rightsquigarrow x \cdot y = f_1 \cdot f_2$

$\mathcal{A}_{\hat{\Sigma}} \xrightarrow{\pi} \mathcal{A}_\Sigma \quad \rightsquigarrow \Phi(x \cdot y) = \pi_*\left(\Phi(f_1) \cdot \Phi(f_2)\right)$

$\mathcal{A}_{\hat{\Sigma}} = [\mathrm{Bl}_0 \mathbb{A}^2 / \mathbb{G}_m^2 \rtimes \mathbb{Z}/2\mathbb{Z}]$, $\mathcal{A}_\Sigma = [\mathbb{A}^2 / \mathbb{G}_m^2 \rtimes \mathbb{Z}/2\mathbb{Z}]$

$\overset{\text{Exercise}}{=} [\{0\} / \mathbb{G}_m^2 \rtimes \mathbb{Z}/2\mathbb{Z}]$ for $\{0\} \subseteq \mathbb{A}^2$.

In fact: whole inters. theory of $\mathcal{A}_\Sigma$ determined by s.p.poly's

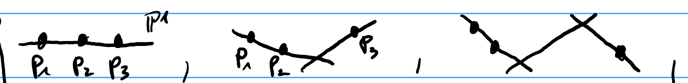
**Thm [MPS]**

$$\Phi : sPP^*(\Sigma) \longrightarrow CH^*(\mathcal{A}_\Sigma) \text{ is an isomorphism.}$$

**Proof idea** Stratification of $\mathcal{A}_\Sigma$ by stacks $B\,\mathbb{G}_m^a \rtimes G$ & higher Chow gps. #

fin. gp.

**Application** Oesinghaus [O] studied stack of expansions

$\mathcal{T} = \{$  $, \ldots \} \subseteq \mathfrak{M}_{0,3}$

moduli of prestable genus 0 curves, 3 markings

**Fact** $\mathcal{T}$ is (non-finite type) Artin fan $\mathcal{A}_{\Sigma_\mathcal{T}}$ where

$\Sigma_\mathcal{T} = \{ \sigma_d = \mathbb{R}^d_{\geq 0} : d \geq 0 \}$ and face maps $\sigma_d \to \sigma_e$ are order-preserving inclusions of coordinates

e.g. $\sigma_2 \rightrightarrows \sigma_3$

$(x_1, x_2) \mapsto (x_1, x_2, 0)$
$(x_1, 0, x_2)$
$(0, x_1, x_2)$

**Thm [O]** $CH^*(\mathcal{T}) = QSym = sPP^*(\Sigma_\mathcal{T})$ ⇝ [MPS] gives second proof.

ring of quasi-symmetric functions

**In geometric situation** $(X, D) \rightsquigarrow \Sigma_{(X,D)}$ and $X \xrightarrow[\text{smooth}]{\tau} \mathcal{A}_{(X,D)}$

**Def**

$$\begin{tikzcd} sPP^*(\Sigma_{(X,D)}) \arrow[rr, "\Phi"] \arrow[dr, "\Phi_{\Sigma(X,D)}"'] & & CH^*(X) \\ & CH^*(\mathcal{A}_{(X,D)}) \arrow[ur, "\tau^*"'] & \end{tikzcd}$$

## Upgrade to piecewise polynomials

$f \in PP^*(\Sigma_{(X,D)}) \rightsquigarrow \exists\, \widehat{\Sigma} \to \Sigma$ st. $f \in sPP^*(\widehat{\Sigma})$

$$\rightsquigarrow \begin{array}{ccc} \widehat{X} & \longrightarrow & \mathcal{A}_{\widehat{\Sigma}} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{A}_{\Sigma_{(X,D)}} \end{array}$$

**Def**

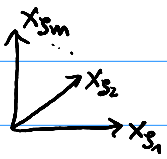
$$\Phi(f) = [\widehat{X}, \Phi_{\widehat{\Sigma}}(f)] \in \log CH^*(X,D)$$

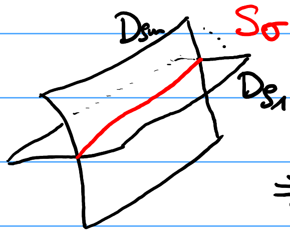
log blow-up — class on $\widehat{X}$

$\rightsquigarrow \Phi : PP^*(\Sigma) \longrightarrow \log CH^*(X,D)$ ring morphism.

**Q** (Geometric) interpretation of $\Phi$ $\boxed{\text{Question 3 for } (X,D)}$

$\widehat{\Sigma} \to \Sigma$ sufficiently fine, $f \in sPP^*(\widehat{\Sigma})$

$\sigma \in \widehat{\Sigma}$ 

$\rightsquigarrow$ 


strata closure $\overline{S_\sigma} \xhookrightarrow{i_\sigma} \widehat{X}$

$\Rightarrow N_{\overline{S_\sigma}/\widehat{X}} = \bigoplus_{i=1}^{m} \mathcal{O}(D_{s_i})$

normal bundle

$f|_\sigma = c_e \cdot x_{s_1}^{e_1} x_{s_2}^{e_2} \cdots x_{s_m}^{e_m} + \cdots \rightsquigarrow \Phi(f) = c_e (i_\sigma)_* \left( c_1(\mathcal{O}_{D_{s_1}})^{e_1 - 1} \cdots c_1(\mathcal{O}_{D_{s_m}})^{e_m - 1} \right) + \cdots$

$\in \mathbb{Q}$ — assume: $e_i \geq 1 \ \forall i$

strata class decorated by Chern classes of normal directions.

**Thm** [HMPPS, Sect. 6]

General recipe for converting

$f \in sPP(\Sigma_X) \rightsquigarrow \Phi(f) = \left( \text{sum of normally decorated strata classes} \right)$

## §4. Applications to moduli spaces of curves & cheesy Star Wars puns

Recall moduli space of stable curves

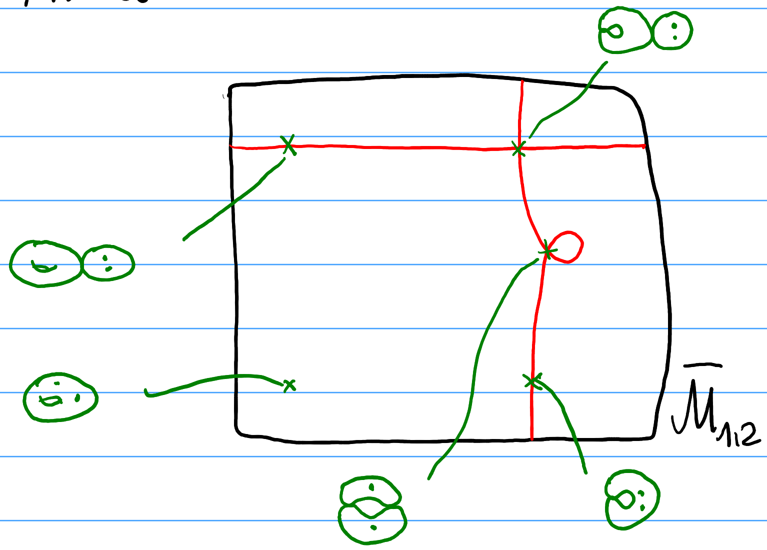
$$\overline{\mathcal{M}}_{g,n} = \{ (C, p_1, \dots, p_n) : \# \mathrm{Aut}(C, p_1, \dots, p_n) < 0 \}$$

at worst nodal curve, a. genus $g$ — $p_i \in C$ distinct smooth pts

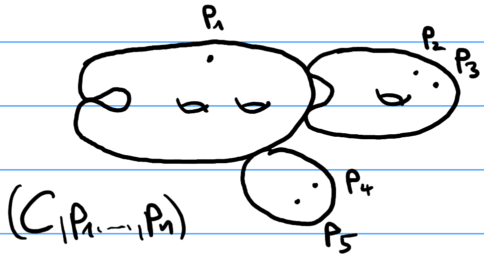
$\rightsquigarrow$ smooth compact orbifold / DM-stack, $\partial \overline{\mathcal{M}}_{g,n}$ nc divisor

$\{(C, p_1, \dots, p_n) : C \text{ singular}\}$

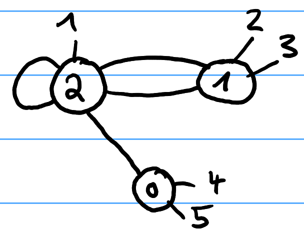
Exa $g = 1$, $n = 2$




Stable graphs describe possible shapes of $(C, p_1, \dots, p_n)$




$\rightsquigarrow$



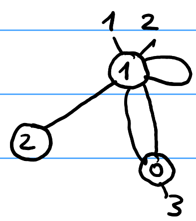

Cone stack $(X, D) = (\overline{\mathcal{M}}_{g,n}, \partial \overline{\mathcal{M}}_{g,n}) \rightsquigarrow \Sigma_{(X,D)} = ?$

Reference: [CCUW]

Strata $S_\Gamma$ of $\overline{\mathcal{M}}_{g,n}$

$\updownarrow$

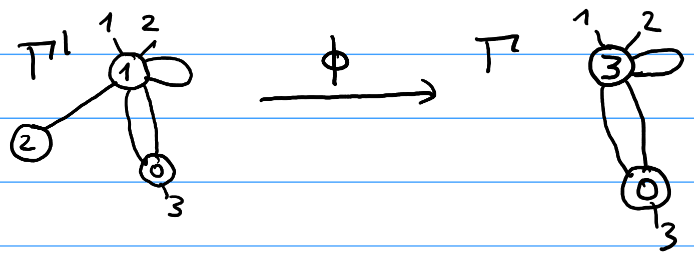
$\mathcal{G}_{g,n} = \{ \text{stable graphs } \Gamma \text{ of genus } g \text{ w/ } n \text{ legs} \}$



Inclusion $S_{\Gamma'} \subseteq \overline{S_{\Gamma}}$

$\Updownarrow$

$\exists$ morphism $\Gamma' \xrightarrow{\phi} \Gamma$ $\hat{=}$ edge contraction ($\rightsquigarrow \phi_E : E(\Gamma) \hookrightarrow E(\Gamma')$)



$\rightsquigarrow \Sigma_{g,n} : \mathcal{G}_{g,n}^{op} \longrightarrow RPC^f$ moduli stack of tropical curves

$\Gamma \longmapsto \sigma_\Gamma = (\mathbb{R}_{\geq 0})^{E(\Gamma)}$

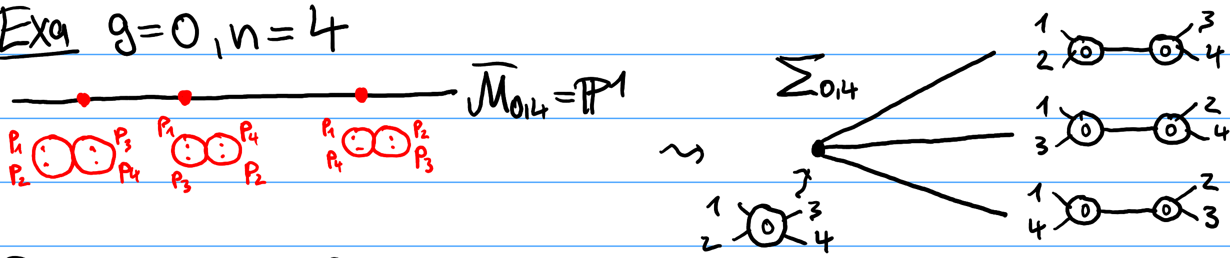
elements of $\sigma_\Gamma$ = stable graphs w/ edge lengths $\ell_e$ ($e \in E(\Gamma)$) tropical curves

$$(\Gamma' \xrightarrow{\phi} \Gamma) \longmapsto \left( \begin{array}{ccc} \sigma_\Gamma & \xrightarrow{i_\phi} & \sigma_{\Gamma'} \\ (\ell_e)_{e \in E(\Gamma)} & \longmapsto & (\ell_{e'})_{e' \in E(\Gamma')} \end{array} \right)$$

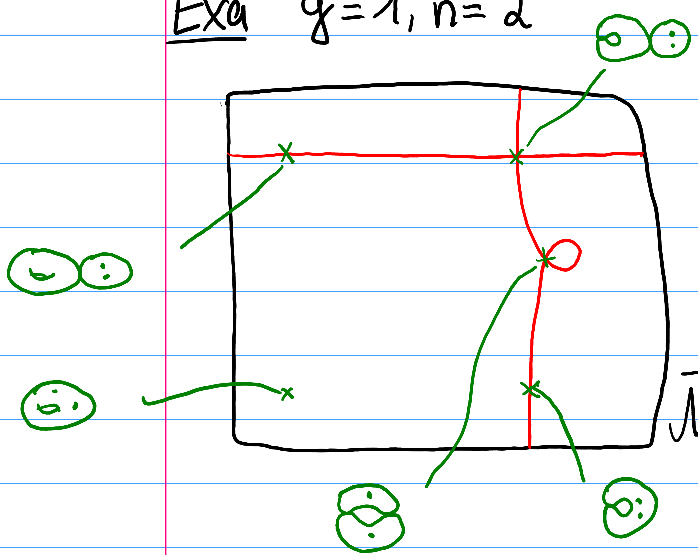
$\hookrightarrow \begin{cases} \ell_e & \text{if } e' = \phi(e) \\ 0 & \text{otherw.} \end{cases}$

Exa $g=0, n=4$

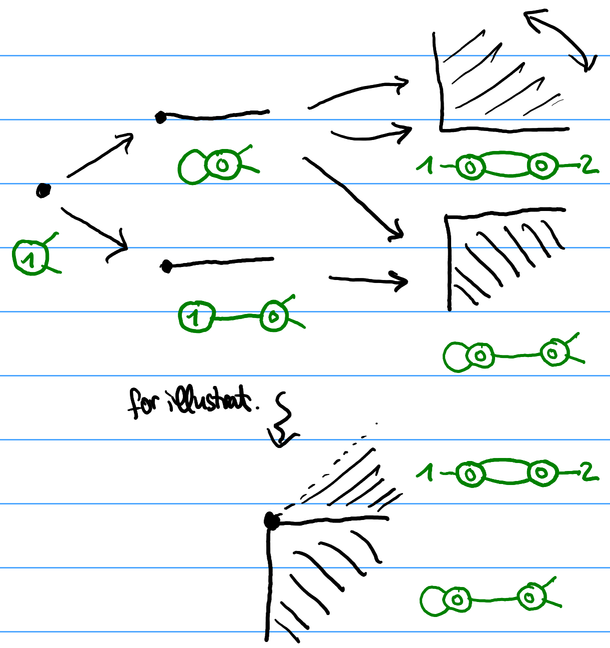
$\overline{\mathcal{M}}_{0,4} = \mathbb{P}^1$ $\rightsquigarrow$ $\Sigma_{0,4}$



Exa $g=1, n=2$



$\overline{\mathcal{M}}_{1,2}$



$\Sigma_{1,2}$

for illustrat.

$\boxed{\text{The phantom automorphism menace}}$

Does $\updownarrow$ count as an automorphism?

$\rightsquigarrow$ Morally yes, but actually no for $\Sigma_{\overline{\mathcal{M}}_{g,n}}$.

# Log tautological rings of $\overline{\mathcal{M}}_{g,n}$

Want notion of tautological class in $\log CH^*(\overline{\mathcal{M}}_{g,n})$.

<u>Def</u> (Attack of the cones)

$$\log R^*_{PP}(\overline{\mathcal{M}}_{g,n}) = \Phi(PP^*(\Sigma_{g,n})) \subseteq \log CH^*(\overline{\mathcal{M}}_{g,n}).$$

<u>Q</u> How much of $\log CH^*$ does this cover?

<u>A</u> We get $\log DR_g(0) = (-1)^g \lambda_g$, and everything in case $g=0$!

<u>Genus zero</u> What is $(\log) CH^*(\overline{\mathcal{M}}_{0,n})$?

boundary divisors $= D_{A \sqcup B}$ for $A \sqcup B = \{1,\dots,n\}$, $|A|,|B| \geq 2$


<u>Exa</u> · $\overline{\mathcal{M}}_{0,3} = pt \rightsquigarrow CH^*(\overline{\mathcal{M}}_{0,3}) = \mathbb{Q} \cdot [\overline{\mathcal{M}}_{0,3}]$

· $\overline{\mathcal{M}}_{0,4} = \mathbb{P}^1 \rightsquigarrow \underbrace{[D_{\{1,2\}\sqcup\{3,4\}}] = [D_{\{1,3\}\sqcup\{2,4\}}] = [D_{\{1,4\}\sqcup\{2,3\}}]}_{WDVV_{0,4}} = [pt] \in CH^1(\mathbb{P}^1)$

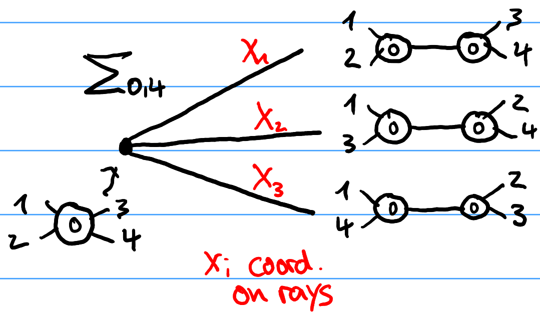
$$CH^*(\overline{\mathcal{M}}_{0,4}) = \mathbb{Q}[D]/(D^2)$$

$\rightsquigarrow$ come from $sPP^*(\overline{\mathcal{M}}_{0,4})$:

$$WDVV^{sPP}_{0,4} = \langle X_1 - X_2, X_2 - X_3 \rangle$$

$$\Downarrow$$

$$WDVV_{0,4} = \Phi(WDVV^{sPP}_{0,4})$$




· For $n \geq 5$ let $\pi^I : \overline{\mathcal{M}}_{0,n} \longrightarrow \overline{\mathcal{M}}_{0,4}$ be forgetful map rememb. only markings $I \subseteq \{1,\dots,n\}$, $|I| = 4$

$$WDVV_{0,n} = \langle (\pi^I)^* WDVV_{0,4} : I \rangle \subseteq \langle [D_{A \sqcup B}] : A \sqcup B = \{1,\dots,n\} \rangle$$

span of all possible pullbacks of WDVV

Similar:

$$WDVV^{sPP}_{0,n} \subseteq sPP^1(\Sigma_{0,n}). \quad \text{from } \pi^I_{trop} : \Sigma_{0,n} \longrightarrow \Sigma_{0,4}$$

<u>Thm [Keel]</u>

$$CH^*(\overline{\mathcal{M}}_{0,n}) = \mathbb{Q}[D_{A \sqcup B} : A \sqcup B \text{ as above}] \Big/ \left( \begin{array}{l} D_{A_1 \sqcup B_1} \cdot D_{A_2 \sqcup B_2} = 0 \text{ if divisors disjoint} \\ WDVV_{0,n} \end{array} \right)$$

## Idea of pf

**Fact** $\overline{\mathcal{M}}_{0,n}$ is an iterated blow-up of $(\mathbb{P}^1)^{n-3}$ along smooth centers

[Fulton] $Z \subseteq X$ smooth closed, $\tilde{X} = Bl_Z X \longrightarrow X$ w/ exc. div. $E \subseteq \tilde{X}$

$\Rightarrow$ $\exists$ exact sequence

$$0 \to CH^*(Z) \longrightarrow CH^*(X) \oplus CH^*(E) \longrightarrow CH^*(\tilde{X}) \longrightarrow 0. \quad (\star)$$

$\rightsquigarrow$ Keel shows Thm by careful comb. analysis. $\square$

**Cor** $CH^*(\overline{\mathcal{M}}_{0,n}) = sPP^*(\Sigma_{0,n}) / (WDVV^{sPP}_{0,n})$.

**Pf**

$D_{A_1 \sqcup B_1} \cdot D_{A_2 \sqcup B_2} = 0$ when divisors disjoint

$\hat{=}$ $\varphi_{D_{A_1\sqcup B_1}}$ and $\varphi_{D_{A_2\sqcup B_2}}$ have disjoint support, so $\varphi_{D_{A_1\sqcup B_1}} \cdot \varphi_{D_{A_2\sqcup B_2}} = 0$.

Stanley-Reisner presentation of $sPP^*$: these are all formal relations between $\varphi_{D_{A\sqcup B}}$

$\Rightarrow$ conclude using [Keel]. $\square$

**Thm** (Pandharipande-Ranganathan-S.-Spelier)

$$\log CH^*(\overline{\mathcal{M}}_{0,n}) = PP^*(\Sigma_{0,n}) / (WDVV^{sPP}_{0,n}).$$

## Idea of proof (Return of the tori)

Construction of [Kapranov] $\rightsquigarrow$ $\overline{\mathcal{M}}_{0,n}$ = Chow quot. of $G(2,n)$ by $\mathbb{G}_m^n/\mathbb{G}_m$ $\ni \begin{pmatrix} p_1 & p_2 & \cdots & p_n \\ q_1 & q_2 & \cdots & q_n \end{pmatrix} \curvearrowleft GL_2$

$\Rightarrow$ $\exists$ smooth q.proj. toric variety $X_{0,n}$ with torus $T$:

$$\begin{tikzcd}
\overline{\mathcal{M}}_{0,n} \arrow[r, hook, "i"] & X_{0,n} \arrow[r] & {[X_{0,n}/T]} \arrow[d, phantom, sloped, "\cong"] \\
G(2,n) \arrow[u, dashed, "/(\mathbb{G}_m^n/\mathbb{G}_m)"] \arrow[r, hook, "\text{Plücker}"] & \mathbb{P}^{\binom{n}{2}-1} \arrow[u, dashed] & \mathcal{A}_{\overline{\mathcal{M}}_{0,n}}
\end{tikzcd}$$

$\rightsquigarrow$ Artin fans of $\overline{\mathcal{M}}_{0,n}$ & $X_{0,n}$ coincide

$\Rightarrow$ $\{\text{log blow-ups } \hat{\mathcal{M}} \to \overline{\mathcal{M}}_{0,n}\} \hat{=} \{\text{Subdiv of } \Sigma_{0,n} = \Sigma_{X_{0,n}}\} \hat{=} \{\text{log blow-ups } \hat{X} \to X_{0,n}\}$

**Check** $i^*$ induces isom. of $CH^*$ on all strata

$\overset{(\star)}{\Longrightarrow}$ remains true for $\hat{i} : \hat{\mathcal{M}} \longrightarrow \hat{X}$.

$$\Rightarrow \log CH^*(\overline{\mathcal{M}}_{0,n}) = \log CH^*(X_{0,n}) \overset{§1}{=} PP^*(\underbrace{\Sigma_{X_{0,n}}}_{=\Sigma_{0,n}}) / (\underbrace{L(\Sigma_{0,n})}_{\overset{\text{Check}}{=} WDVV^{sPP}_{0,n}}) \quad \square$$

**Cor** $\log R^*_{PP}(\overline{\mathcal{M}}_{0,n}) = \log CH^*(\overline{\mathcal{M}}_{0,n})$

**Problem** $\log R_{PP}$ too small in general!

$\to \log DR_g(A) \notin \log R^*_{PP}(\overline{\mathcal{M}}_{g,n})$ (e.g. $g=3, n=3$)

$\to \quad \psi_1 \quad \notin \log R^*_{PP}(\overline{\mathcal{M}}_{2,1})$

Pf $0 \neq \psi_1 \in CH^1(\underbrace{\mathcal{M}_{2,1}}_{\text{smooth curves}})$ but $\log R^*_{PP}(\mathcal{M}_{g,n})$ vanishes in pos. degree!

$\overset{\shortparallel}{\langle \text{normally decor. strata classes} \rangle}$

**Def** (Revenge of the psis)

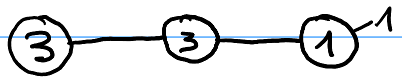
$$\log R^*_{sm}(\overline{\mathcal{M}}_{g,n}) = \mathrm{im}\left( \log R^*_{PP}(\overline{\mathcal{M}}_{g,n}) \otimes R^*(\overline{\mathcal{M}}_{g,n}) \longrightarrow \log CH^*(\overline{\mathcal{M}}_{g,n}) \right).$$

**Thm** [MR, HS, HMPPS]

$$\log DR_g(A) \in \log R^*_{sm}(\overline{\mathcal{M}}_{g,n}).$$

**Q** So $\log R_{sm}$ big enough?

**A** No! Consider stable graph $\Gamma$

$\Gamma$  $\rightsquigarrow$

$\alpha \quad \kappa_1 \otimes 1 \otimes 1$

$$\begin{tikzcd} E \arrow[r, hook, "i_E"] \arrow[d, "\pi_E"] & \widehat{\mathcal{M}} \arrow[d] \\ \overline{S}_\Gamma \arrow[r, hook, "i_\Gamma", "\text{codim } 2"'] & \overline{\mathcal{M}}_{7,1} \end{tikzcd}$$

(essentially) blow up $\overline{S}_\Gamma$

$\alpha \in CH^1(\overline{S}_\Gamma)$

Consider $\widehat{\gamma}_\alpha = (i_E)_*(\pi_E^* \alpha) \in \log CH^2(\overline{\mathcal{M}}_{7,1})$.

"The exceptional divisor strikes back"

**Prop** $\widehat{\gamma}_\alpha \notin \log R^*_{sm}(\overline{\mathcal{M}}_{7,1})$.

Issue: $\widehat{\gamma}_\alpha$ "looks" tautological, but $\log R^*_{sm}$ cannot combine deco. at vertex + blow-up.

**Solution** decorated log strata classes from log gluing pushforwards

**Sketch**

$\xi_\Gamma : \overline{\mathcal{M}}^{st}_\Gamma = \prod \overline{\mathcal{M}}_{g(v), n(v)} \longrightarrow \overline{\mathcal{M}}_{g,n}$ gluing map

put strict log structure

Pim Spelier: $f \in PP_*(\Sigma_{\overline{\mathcal{M}}^{str}_\Gamma}) \rightsquigarrow \Phi(f) \in \log CH_*(\overline{\mathcal{M}}^{str}_\Gamma)$

piecew. poly vanishing on "bdry" of $\Sigma_{\overline{\mathcal{M}}^{str}_\Gamma}$

see [Barrott] for definition

For $\alpha =$ monomial in $\kappa, \psi$-classes on factors $\overline{\mathcal{M}}_{g(v),n(v)}$ of $\overline{\mathcal{M}}_\Gamma$

$$\rightsquigarrow [\Gamma, f, \alpha] = (\xi_\Gamma)_*(\alpha \cap \Phi(f)) \in \log CH^*(\overline{\mathcal{M}}_{g,n}).$$

Def (A new homology)

$$\log R^*(\overline{\mathcal{M}}_{g,n}) = \langle [\Gamma, f, \alpha] : \Gamma, f, \alpha \text{ as above} \rangle_{\mathbb{Q}\text{-lin. span}} \subseteq \log CH^*(\overline{\mathcal{M}}_{g,n})$$

Thm (PRSS, in progress)

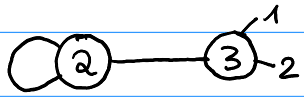
- $\log R^*$ closed under inters. product, expl. formula generalizing [GP]
- $\left.\begin{array}{l}\text{Pixton's conj. on taut. relations of } \overline{\mathcal{M}}_\Gamma \\ \text{\& blow-up formula } (\star)\end{array}\right\}$ determine $\log R^*$ (in principle)
- generalization to arbitrary $(X, D)$ smooth nc pair $\rightsquigarrow$ generating set $[\sigma, f, \alpha]$ of $\log CH^*(X, D)$.

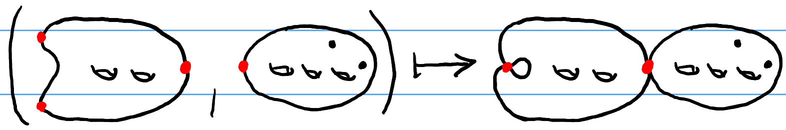
  $\sigma \in \Sigma_{(X,D)}$, $f \in PP_*(\mathrm{Star}_\sigma)$, $\alpha \in CH^*(\overline{S_\sigma})$ more or less

Thanks for your attention!

# Appendix: Tautological classes on $\overline{\mathcal{M}}_{g,n}$

$\Gamma$ stable graph $\rightsquigarrow$ $\xi_\Gamma : \overline{\mathcal{M}}_\Gamma = \prod_{v\in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)} \xrightarrow{\text{gluing map}} \overline{\mathcal{M}}_{g,n}$





**Def** · $\psi_i = c_1(\mathbb{L}_i)$ , $\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n}$ line bdle w/ $\mathbb{L}_i|_{(C,p_1,\dots,p_n)} = T^*_{p_i}C$

· $\kappa_a = \pi_* \psi_{n+1}^{a+1}$ , $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ forgetful map

$(C,p_1,\dots,p_n,p_{n+1}) \mapsto (C,p_1,\dots,p_n)$ when $C$ smooth

**Def** Decorated strata classes

$\alpha$ = product of $\kappa$, $\psi$-classes on factors $\overline{\mathcal{M}}_{g(v),n(v)}$ of $\overline{\mathcal{M}}_\Gamma$

$\rightsquigarrow$ $[\Gamma,\alpha] := (\xi_\Gamma)_* \alpha \in CH^*(\overline{\mathcal{M}}_{g,n})$ dec. strat. class

**Thm/Def** (Tautological ring [GP])

The $\mathbb{Q}$-linear span

$$R^*(\overline{\mathcal{M}}_{g,n}) = \langle [\Gamma,\alpha] : \Gamma \text{ st. graph}, \alpha \text{ decorat.} \rangle \subseteq CH^*(\overline{\mathcal{M}}_{g,n})$$

is closed under intersect. products $\leftarrow$ explicit formula $[\Gamma_1,\alpha_1]\cdot[\Gamma_2,\alpha_2] = \sum_{\Gamma,\dots} [\Gamma, \dots]$

## References


[ACMUW] Abramovich, Chen, Marcus, Ulirsch, Wise – Skeletons and fans of logarithmic structures (2016)

[Barrott] Barrott – Logarithmic Chow theory (2018)

[Brion] Brion – Piecewise polynomial functions, convex polytopes and enumerative geometry (1996)

[CCUW] Cavalieri, Chan, Ulirsch, Wise – A moduli stack of tropical curves (2020)

[CGP] Chan, Galatius, Payne – Tropical curves, graph complexes and top weight cohomology of $\mathcal{M}_g$ (2021)

[CLS] Cox, Little, Schenk – Toric varieties

[CMR] Cavalieri, Markwig, Ranganathan – Pluricanonical cycles and tropical covers (2022)

[EG] Edidin, Graham – Equivariant intersection theory (1998)

[Fantechi] Fantechi – Stacks for everybody (2001)

[Fulton] Fulton – Intersection theory

[GP] Graber, Pandharipande – Constructions of nontautological classes on moduli spaces of curves (2003)

[H] Holmes – Extending the double ramific. cycle by resolving the Abel–Jacobi map (2019)

[HMPPS] Holmes, Molcho, Pandharipande, Pixton, Schmitt – Logarithmic double ramification cycles (2022)

[HPS] Holmes, Pixton, Schmitt – Multiplicativity of the double ramification cycle (2019)

[HS] Holmes, Schwarz – Logarithmic intersection theory of double ramification cycles (2021)

[Kapranov] Kapranov – Chow quotients of Grassmannians (1993)

[K] Kresch – Cycle groups for Artin stacks (1999)

[MPS] Molcho, Pandharipande, Schmitt – The Hodge bundle, the universal 0-section and $\log CH^*(\overline{\mathcal{M}}_{g,n})$ (2023)

[O] Oesinghaus – Quasisymmetric functions and the Chow ring of the stack of expanded pairs (2019)

[Payne] Payne – Equivariant Chow cohomology of toric varieties (2006)

[U] Ulirsch – Non-Archimedean geometry of Artin fans (2019)