1 Introduction

The moduli space of stable curves $\overline{M}_{g,n}$ is a fundamental object in algebraic geometry, parametrizing isomorphism classes of n-pointed algebraic curves of genus g with only mild singularities (nodes). Its construction and basic properties were first established by Deligne and Mumford in 1969, who proved in particular that $\overline{M}_g$ is irreducible [DM69]. Since then, $\overline{M}_{g,n}$ has been extensively studied from various perspectives [HM98]. One fruitful approach is to investigate its cohomology and Chow rings. In this vein, Mumford initiated the study of certain natural cohomology classes on $\overline{M}_{g,n}$ [Mum83]. The subring of the Chow ring generated by these canonical classes is now known as the *tautological ring* of the moduli space.

The tautological ring $R^*(\overline{M}_{g,n})$ is generated by tautological classes such as the psi classes ψ_i (first Chern classes of the line bundles associated to marked points), kappa classes κ_j (descending pushforwards of powers of ψ-classes), the lambda classes λ_i (Chern classes of the rank g Hodge bundle), and classes of boundary strata corresponding to nodal curve degenerations [Mum83, Fab99]. These classes capture the most geometrically natural subspace of the moduli's Chow ring. Faber famously conjectured a precise structure for $R^*(\overline{M}_g)$, predicting that it behaves like a Gorenstein algebra of dimension $g-2$ [Fab99]. In particular, his conjectures imply that all tautological classes in degree above $g-2$ vanish and that there is a form of Poincaré duality pairing on the tautological subring. These conjectures have been verified in low-genus cases and are supported by various computations and partial results [Fab99, FL99], but remain open in general. Moreover, tautological intersections underlie several remarkable results in enumerative geometry. Witten's conjecture—proved by Kontsevich's landmark work on intersection theory—asserts that a generating function for ψ-class intersection numbers on $\overline{M}_{g,n}$ satisfies the KdV integrable hierarchy [Kon92]. Another example is the ELSV formula of Ekedahl–Lando–Shapiro–Vainshtein, which expresses Hurwitz numbers (counts of branched covers of curves) as integrals of ψ and λ classes on $\overline{M}_{g,n}$ [ELSV01]. These connections highlight the rich combinatorial and physical significance of the tautological ring.

Recent years have witnessed further breakthroughs in our understanding of tautological classes. A celebrated example is the proof of Mumford's conjecture on the stable cohomology of moduli spaces: Madsen and Weiss showed that in the limit of large genus, the entire rational cohomology ring is generated by the κ-classes [MW07]. In other words, beyond a certain range, no new cohomology classes appear outside the tautological subring. [Note by Johannes: This is true for the tautological ring of the moduli space M_g of smooth curves with no marked points, but *not* for $\overline{M}_{g,n}$!] This deep result, achieved through topological methods, confirms that tautological classes indeed capture the full stable cohomology. Nonetheless, for fixed genus g, the structure of the tautological ring $R^*(\overline{M}_g)$ remains intricate and is the subject of ongoing research. Understanding the relations and symmetries in this ring is crucial for a deeper grasp of the geometry of $\overline{M}_{g,n}$, and it continues to be an area of active interplay between algebraic geometry, topology, and combinatorics.

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