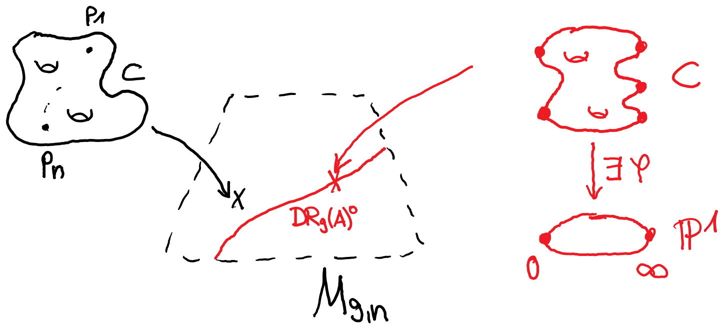
Logarithmic double ramification cycles

Workshop "Integrability, enumerative geometry and quantisation"
Stony Brook, August 2022
Johannes Schmitt (j.w. Holmes, Molcho, Pandharipande, Pixton)

§1. Double ramification cycles

Let $g,n \geq 0$ with $2g-2+n>0$, $A=(a_1,\ldots,a_n) \in \mathbb{Z}^n$ with $\sum_{i=1}^n a_i = 0$.

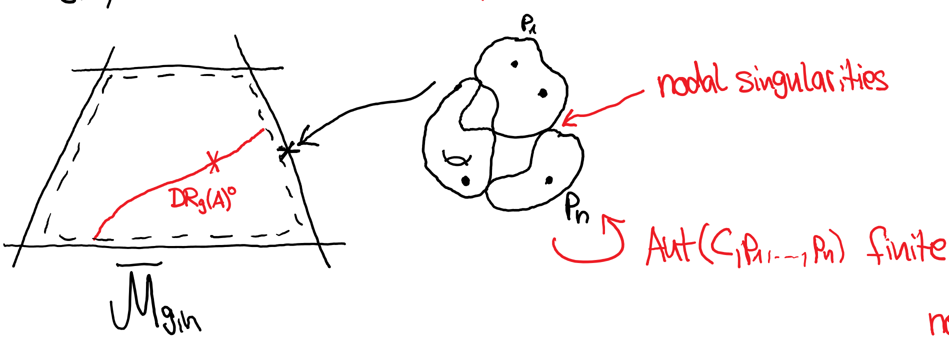


s.t. $\mathrm{div}(\varphi) = \sum_{i=1}^n a_i p_i$

moduli space of smooth genus g curves with n marked points

Equivalently $DR_g(A)^\circ = \{(C,p_1,\ldots,p_n) \mid \mathcal{O}_C(\sum a_i p_i) \cong \mathcal{O}_C\}$ double ramification locus

Q How to extend $DR_g(A)^\circ$ (and $[DR_g(A)^\circ]$) to compactification $\overline{\mathcal{M}}_{g,n}$, the moduli space of stable curves?



not closed!

Problem: $\{(C,p_1,\ldots,p_n) \in \overline{\mathcal{M}}_{g,n} \mid \mathcal{O}_C(\sum a_i p_i) \cong \mathcal{O}_C\} \subseteq \overline{\mathcal{M}}_{g,n}$

Reason: equality in $\mathrm{Pic}^0_{g,n} = \{(C,p_1,\ldots,p_n,\mathcal{L}) \mid \begin{smallmatrix}\mathcal{L}/C \text{ line bdle} \\ \deg(\mathcal{L})=0\end{smallmatrix}\}$

not separated!

Solution Try to formulate equality in

$\mathrm{Pic}^{\underline{0}}_{g,n} = \{(C,p_1,\ldots,p_n,\mathcal{L}) \mid \begin{smallmatrix}\mathcal{L}/C \text{ line bundle}, \\ \deg_{C'}(\mathcal{L})=0 \ \forall C' \subseteq C \text{ component}\end{smallmatrix}\} \subseteq \mathrm{Pic}^0_{g,n}$

this is separated

Consider diagram:

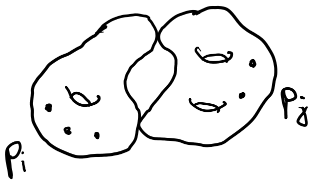
$\mathrm{Pic}^{\underline{0}}$

Consider diagram:

$$\begin{tikzcd} \mathrm{Pic}^0_{g,n} \arrow[d] \\ \overline{\mathcal{M}}_{g,n} \arrow[u, bend left, "s_0"', "\mathcal{L}=\mathcal{O}"] \arrow[u, bend right, "s_A", "\mathcal{L}=\mathcal{O}_C(\sum a_i p_i)"'] \end{tikzcd}$$

$\rightsquigarrow$ Would like: $DR_g(A) = S_0 \cap S_A$

<u>New problem</u> S_A is only rational map:



$\rightsquigarrow \mathcal{L} = \mathcal{O}_C(\sum a_i p_i)$ not multidegree 0

<u>Solution</u> Resolve indeterminacy of S_A using blowups
[Holmes '17]

$$\begin{tikzcd} \mathrm{Pic}^0_{g,n} \arrow[d] & & \\ \overline{\mathcal{M}}_{g,n} \arrow[u, bend left, "s_0"', "\mathcal{L}=\mathcal{O}"] \arrow[u, bend right, "s_A", "\mathcal{L}=\mathcal{O}_C(\sum a_i p_i)"'] & \overline{\mathcal{M}}^{\diamond}_{g,A} \arrow[l, "q"', "\text{log-blowup}"] & \mathcal{M}^{\diamond}_{g,A} \arrow[l, hook', "i"', "\text{open}"] \arrow[ull, bend right, "\hat{s}_0"] \arrow[ull, bend right=40, "\hat{s}_A"'] \\ & & DRL_A = \hat{s}_0 \cap \hat{s}_A \arrow[u, phantom, "\cup\!\!\mid"] \arrow[ul, "i'"] \end{tikzcd}$$

iterated blowup of boundary strata (see § 2)

proper

$\rightsquigarrow$ double ramification locus: $\quad (q \circ i')(DRL_A) \subseteq \overline{\mathcal{M}}_{g,n}$

double ramification cycle: $\quad DR_g(A) = (q \circ i')_* \big(\underbrace{\hat{s}_A^* [\mathrm{im}(\hat{s}_0)]}_{\text{cycle supported on } DRL_A} \big) \in CH^g(\overline{\mathcal{M}}_{g,n})$

Chow ring

Generalization for $\omega_C^{\otimes k} \cong \mathcal{O}(\sum a_i p_i)$
(Sauvaget, Costantini-Sauvaget-S.,
Costantini-Möller-Zachhuber)

Formula in tautological ring
$R^*(\overline{\mathcal{M}}_{g,n}) \subseteq CH^*(\overline{\mathcal{M}}_{g,n})$
(Janda-Pandharipande-Pixton-Zvonkine,
Holmes-S., Bae-Holmes-Pandharipande-
S.-Schwarz)

DR cycles

Integrable systems
(Buryak-Rossi)

Gromov-Witten theory
(Fan-Wu-You, Oberdieck-Pixton)

However, we get something better above:

$$\begin{tikzcd} \mathrm{Pic}^0_{g,n} & & \arrow[ll, bend right, "\hat{s}_A"'] \arrow[ll, bend left, "\hat{s}_0"] \end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Pic}^0_{g,n} \arrow[d] & & \\
\overline{\mathcal{M}}_{g,n} \arrow[u, bend left, "s_0"', "\mathcal{L}=\mathcal{O}_C"] \arrow[u, bend right, "s_A", "\mathcal{L}=\mathcal{O}_C(\sum a_i p_i)"'] & \overline{\mathcal{M}}^{\diamond}_{g,A} \arrow[l, "q"', "\text{log-blowup}"] & \mathcal{M}^{\diamond}_{g,A} \arrow[l, "i"', "\text{open"] \arrow[ull, bend right, "\hat{s}_A"'] \arrow[ull, bend right=15, "\hat{s}_0"] \\
& & \mathrm{DRL}_A = \hat{s}_0 \cap \hat{s}_A \arrow[u, phantom, "\cup"] \arrow[ul, "i'"]
\end{tikzcd}$$

$$\widehat{\mathrm{DR}}_g(A) = i'_*\left(\hat{s}_A^*[\mathrm{im}(\hat{s}_0)]\right) \in CH^g(\overline{\mathcal{M}}^{\diamond}_{g,A})$$

$\rightsquigarrow$ $\mathrm{DR}_g(A) = q_* \widehat{\mathrm{DR}}_g(A)$ $\rightsquigarrow$ refinement

One issue: ambient space $\overline{\mathcal{M}}^{\diamond}_{g,A}$ depends on A & not canonical.

Def (logarithmic Chow ring)

$$\log CH^*(\overline{\mathcal{M}}_{g,n}) = \varinjlim_{\substack{\widehat{\mathcal{M}} \to \overline{\mathcal{M}}_{g,n} \\ \text{log-blowup}}} CH^*(\widehat{\mathcal{M}})$$

$$= \left\{ \left(\widehat{\mathcal{M}} \to \overline{\mathcal{M}}_{g,n}, \alpha \in CH^*(\widehat{\mathcal{M}}) \right) \right\} \Big/ \begin{array}{l} (\widehat{\mathcal{M}},\alpha) \sim (\widehat{\widehat{\mathcal{M}}},\beta) \\ \text{for } \widehat{\widehat{\mathcal{M}}} \xrightarrow{\pi} \widehat{\mathcal{M}} \to \overline{\mathcal{M}}_{g,n} \\ \text{w/ } \pi^*\alpha = \beta \end{array}$$

Slogan
Describe intersection theory of all log-blowups of $\overline{\mathcal{M}}_{g,n}$ simultaneously

Thm/Def (Holmes)
The cycle $\log \mathrm{DR}_g(A) = [\overline{\mathcal{M}}^{\diamond}_{g,A}, \widehat{\mathrm{DR}}_g(A)] \in \log CH^g(\overline{\mathcal{M}}_{g,n})$
is independent on the choice of $\overline{\mathcal{M}}^{\diamond}_{g,A}$.
$\rightsquigarrow$ logarithmic double ramification cycle

Advantages

Holmes-Pixton-S.'17:

$$\log \mathrm{DR}_g(A) \cdot \log \mathrm{DR}_g(B) = \log \mathrm{DR}_g(A) \cdot \log \mathrm{DR}_g(A+B)$$

Cavalieri-Markwig-Ranganathan '22:
Reconstruct double Hurwitz numbers from $\log \mathrm{DR}_g(A)$

⇝ false / not known for $DR_g(A)$!

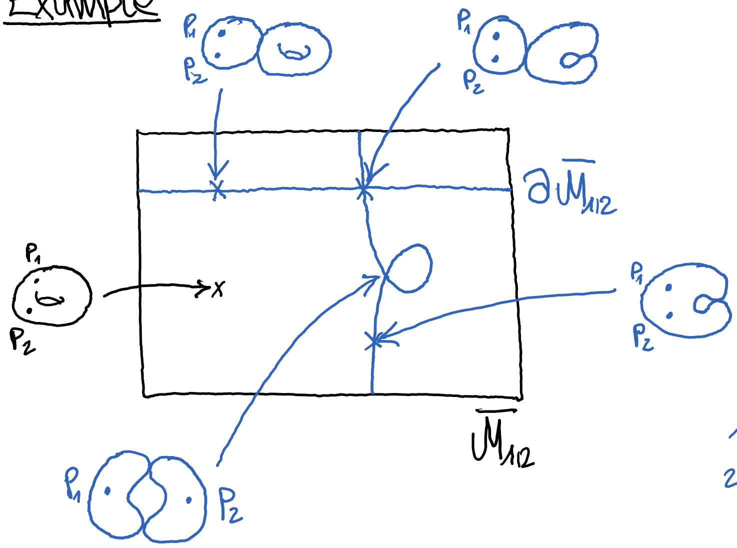
In recent paper w/ Holmes, Molcho, Pandharipande, Pixton we compute $\log DR_g(A) \in \log CH^*(\overline{\mathcal{M}}_{g,n})$ explicitly.

Goals for rest of talk

→ develop language & state (shape of the) result
→ explain ideas going into the proof

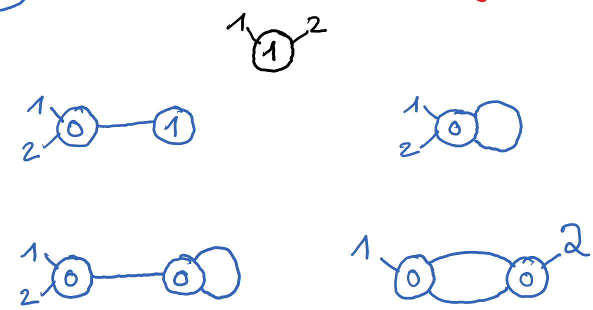
§2. Log blowups & subdivisions of $M_{g,n}^{trop}$

Example

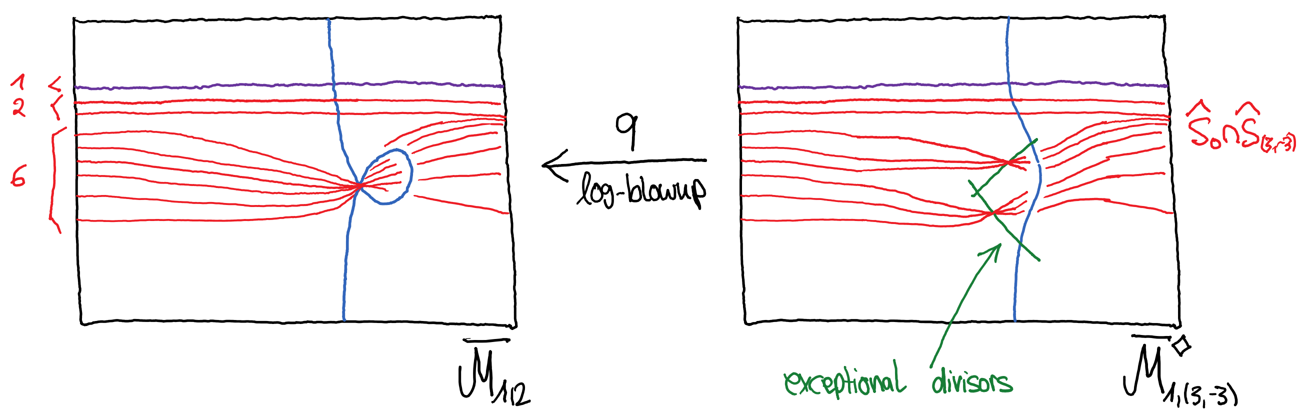


Strata of $\overline{\mathcal{M}}_{1,2}$
≙ loc. closed subsets where shape of curve is fixed.

stable graph



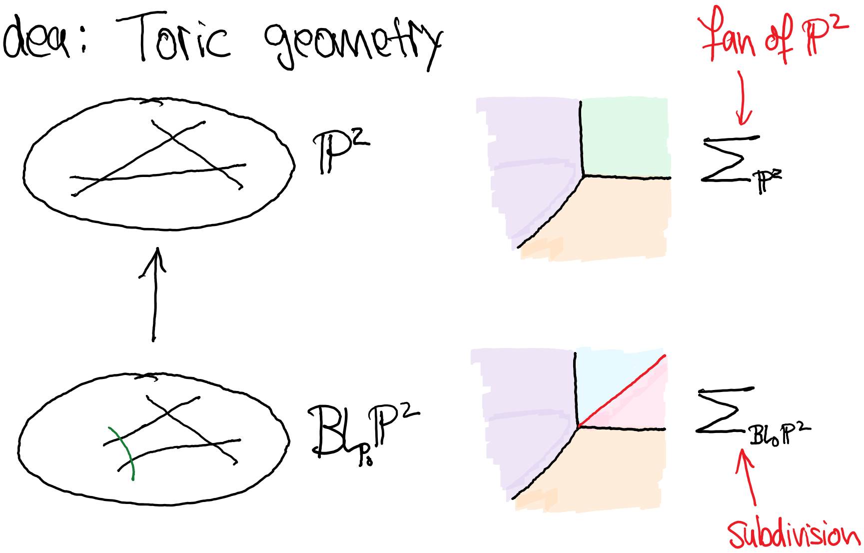
$DR_1(3,-3) \rightsquigarrow C$ smooth: $\mathcal{O}_C(3P_1 - 3P_2) \cong \mathcal{O} \Leftrightarrow P_2$ is 3-torsion in (C, P_1)



Q How can we describe these log-blowups efficiently?

A Tropical geometry !?

Idea: Toric geometry

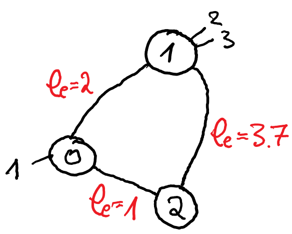


log-blowups of X $\overset{1:1}{\updownarrow}$ subdivisions of Σ_X

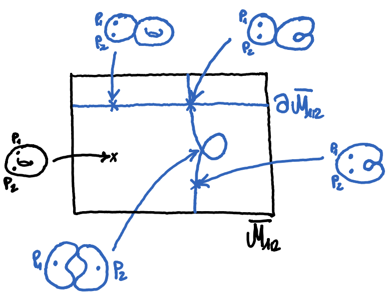
In situation above:

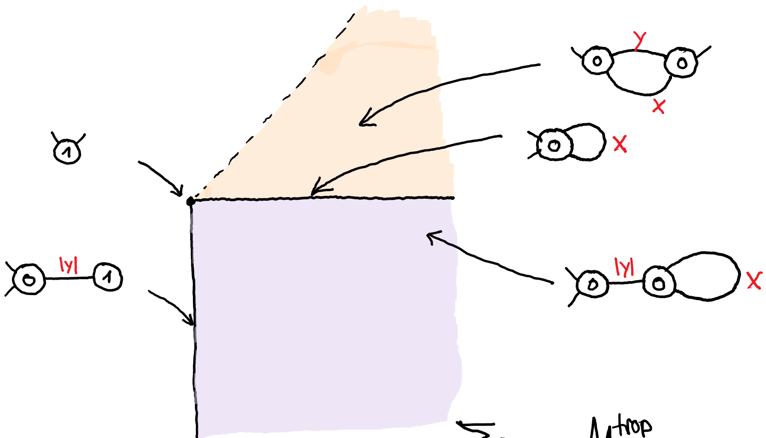
$$\Sigma_{\overline{\mathcal{M}}_{g,n}} = M_{g,n}^{trop} \quad \text{moduli space of tropical curves}$$

Tropical curve = Stable graph Γ + lengths $\ell_e \in \mathbb{R}_{>0}$ for $e \in Edges(\Gamma)$



Fixing g,n, such trop. curves are parameterized by a nice moduli space, which is a cone stack / stacky fan.





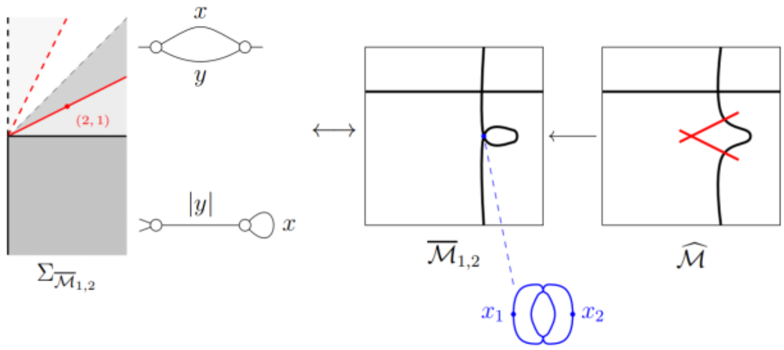
$$\Sigma_{\overline{\mathcal{M}}_{1,2}} = M_{1,2}^{trop}$$

Basically: $\Sigma_{\overline{\mathcal{M}}_{g,n}} = \bigsqcup_{\Gamma \text{ stable graph}} \mathbb{R}_{\geq 0}^{E(\Gamma)} / \sim$ ← edge contractions & automorphisms

$\sigma_\Gamma = \{E(\Gamma) \xrightarrow{\ell} \mathbb{R}_{\geq 0}\}$

Proposition

log-blowups $\widehat{\mathcal{M}}$ of $\overline{\mathcal{M}}_{g,n}$ $\xleftrightarrow{1:1}$ subdivisions $\widehat{\Sigma}$ of $\Sigma_{\overline{\mathcal{M}}_{g,n}}$



Next: Given $\widehat{\mathcal{M}} \to \overline{\mathcal{M}}_{g,n}$, how do we describe interesting classes in $CH^*(\widehat{\mathcal{M}})$?

§3. Piecewise polynomials and the formula for logDR

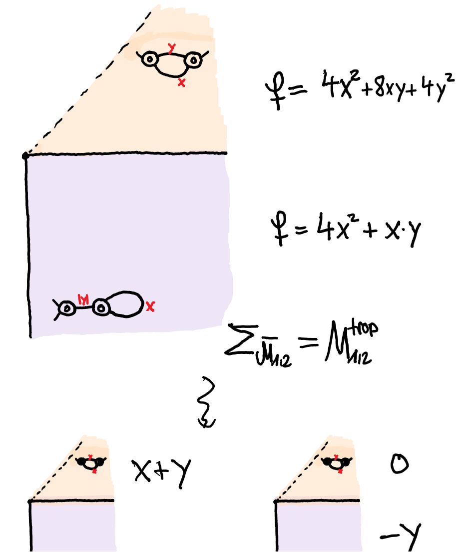
Def Given a subdivision $\widehat{\Sigma}$ of $\Sigma_{\overline{\mathcal{M}}_{g,n}}$, a strict piecewise polynomial is a continuous function

$$\varphi : \widehat{\Sigma} \longrightarrow \mathbb{R}$$

which is a polynomial in the edge lengths ℓ_e on each cone $\sigma \in \widehat{\Sigma}$. A piecewise polynomial is a strict piecewise polynomial on some further subdivision $\hat{\hat{\Sigma}}$ of $\widehat{\Sigma}$.

$\rightsquigarrow$ $sPP(\widehat{\Sigma})$, $PP(\widehat{\Sigma})$.

Example



Theorem (Holmes–Schwarz '21)

There exist unique ring homomorphisms

$$\Phi : sPP(\widehat{\Sigma}) \longrightarrow CH^*(\widehat{\mathcal{M}})$$

satisfying

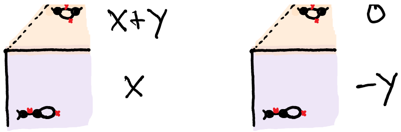
(a) $\underbrace{\Phi(\varphi)}_{\text{p. linear}} = \sum_{\tau \text{ ray of } \widehat{\Sigma}} \varphi(v_\tau) \cdot [D_\tau]$

primitive generator of τ — divisor $D_\tau \subseteq \widehat{\mathcal{M}}$ dual to τ

(b) for $\hat{\hat{\Sigma}} \xrightarrow{p} \widehat{\Sigma}$ corresponding to $\hat{\hat{\mathcal{M}}} \xrightarrow{\pi} \widehat{\mathcal{M}}$:

$$sPP(\widehat{\Sigma}) \xrightarrow{\Phi} CH^*(\widehat{\mathcal{M}})$$

p^* π^*



$$\varphi_{irr} \overset{\Phi}{\mapsto} \delta_{irr} \qquad \varphi_0 \overset{\Phi}{\mapsto} \delta_0$$

$$\varphi = 4\varphi_{irr}^2 - \varphi_{irr}\cdot\varphi_0$$
$$\Rightarrow \Phi(\varphi) = 4\delta_{irr}^2 - \delta_{irr}\cdot\delta_0$$

$$\begin{tikzcd} sPP(\hat{\Sigma}) \arrow[r, "\Phi"] \arrow[d, "p^*"] & CH^*(\widehat{\mathcal{M}}) \arrow[d, "\pi^*"] \\ sPP(\hat{\hat{\Sigma}}) \arrow[r, "\Phi"] & CH^*(\widehat{\widehat{\mathcal{M}}}) \end{tikzcd}.$$

This gives rise to

$$\Phi : PP(\Sigma_{\overline{\mathcal{M}}_{g,n}}) \longrightarrow \log CH^*(\overline{\mathcal{M}}_{g,n}).$$

Now we can state (the shape of) our formula for $\log DR_g(A)$.

<u>Thm</u> (Holmes-Molcho-Pandharipande-Pixton-S. '22)

$c_1(\mathbb{L}_i)$ cotangent line class

Given $g, n, A \in \mathbb{Z}^n$ as before, let $\eta = \sum_{i=1}^n \frac{a_i^2}{2}\psi_i \in CH^1(\overline{\mathcal{M}}_{g,n}) \subseteq \log CH^1(\overline{\mathcal{M}}_{g,n})$

Then there exists an <u>explicit</u> subdivision $\hat{\Sigma}$ of $\Sigma_{\overline{\mathcal{M}}_{g,n}}$ and $\varphi_L \in sPP^1(\hat{\Sigma})$, $\varphi_P \in sPP(\hat{\Sigma})$ such that

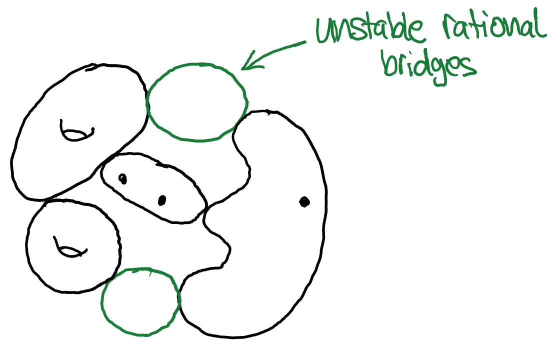
$$\log DR_g(A) = \Big[\exp(\eta + \Phi(\varphi_L)) \cdot \Phi(\varphi_P)\Big]_g \in \log CH^g(\overline{\mathcal{M}}_{g,n}).$$

take codim g part

§4. Idea of proof

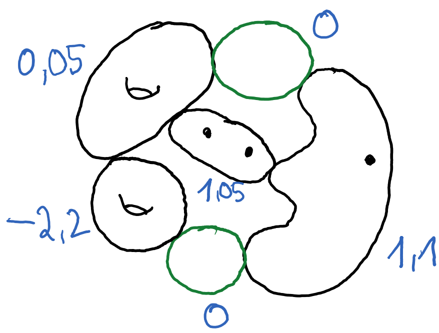
Subdivision $\hat{\Sigma}$ from moduli of stable line bundles

quasi-stable curve



Stability condition θ

= assignment of real numbers $\theta_{C'}$ for • all q.stable C,
• $C' \subseteq C$ component



(Kass-Pagani
Melo-Molcho-Ulirsch
-Viviani)

compatible w/ smoothings

compatible w/ smoothings

Def $\mathcal{L}/C$ admissible (semi)stable line bdle if

- $\deg_{C'} \mathcal{L} = 1$, C' unstable
- $\left|\deg_D \mathcal{L} - \underset{\substack{\uparrow \\ = \sum_{C' \subseteq D} \theta_{C'}}}{\theta_D}\right| \underset{(=)}{<} \frac{\#(D \cap D^c)}{2}$, $D \subsetneq C$ proper subcurve

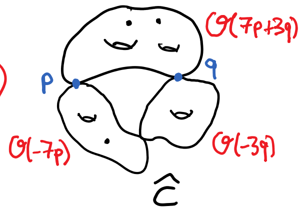
Def θ is called

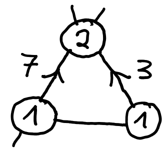
- small, if $\forall \mathcal{L}/C$ stable w/ multideg. θ are adm. stable
- nondegenerate, if stable = semistable

Fix θ small, nondegenerate

$$\overline{\mathcal{M}}^{\theta}_{g,A} = \Big\{ \big(\underset{\substack{\uparrow \\ \text{q. stable}}}{\widehat{C}}, p_1, \dots, p_n, \underset{\substack{\uparrow \\ \text{piecewise linear} \\ \text{function on } \widehat{C} \\ \hat{=}\ \text{twist}}}{\alpha} \big) \;\Big|\; \mathcal{L}(\alpha) = \mathcal{O}_{\widehat{C}}(\textstyle\sum a_i p_i)(\alpha) \text{ adm. stable} \Big\}$$

$\rightsquigarrow \mathcal{O}_{\widehat{C}}(\alpha)$





Thm (HMPPS)

$\pi : \overline{\mathcal{M}}^{\theta}_{g,A} \longrightarrow \overline{\mathcal{M}}_{g,n}$ is log blowup and for

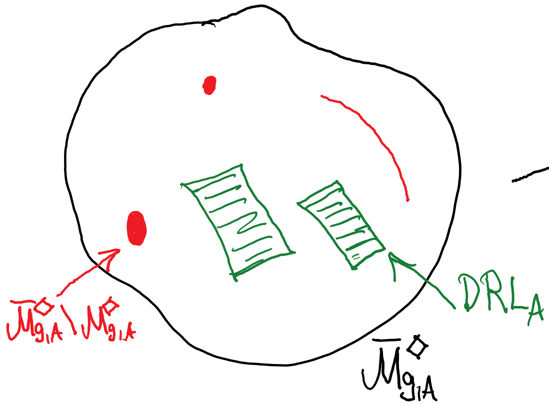
$$\overline{\mathcal{M}}^{\theta}_{g,A} \supseteq \mathcal{M}^{\theta}_{g,A} = \Big\{ (\widehat{C}, p_1, \dots, p_n, \alpha) \;\Big|\; \begin{array}{l} \widehat{C} \text{ stable}, \\ \mathcal{L}(\alpha) \text{ multideg } 0 \end{array} \Big\}$$

we have

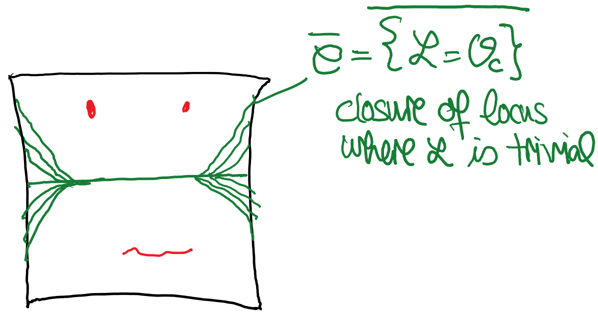
$$\begin{tikzcd}
\mathrm{Pic}^0_{g,n} \arrow[d] & & \\
\overline{\mathcal{M}}_{g,n} & \underset{\overline{\mathcal{M}}^{\diamond}_{g,A}}{\overline{\mathcal{M}}^{\theta}_{g,A}} \arrow[l] \arrow[ul, bend right, "\widehat{S}_A"'] & \underset{\mathcal{M}^{\diamond}_{g,A}}{\mathcal{M}^{\theta}_{g,A}} \arrow[l, hook'] \arrow[ull, bend right, "\widehat{S}_0"] \arrow[ull, bend right=40, "\mathcal{L}(\alpha)"']
\end{tikzcd}$$

$$\boxed{\begin{array}{c} \widehat{\Sigma} \text{ from Thm} \\ \updownarrow \\ \overline{\mathcal{M}}^{\theta}_{g,A} \end{array}}$$

Even better



$$\xrightarrow[S_{\mathcal{L}(\alpha)}]{(\hat{C},\mathcal{L}(\alpha))}$$



$$\mathcal{P}ic_g = \left\{ (C,\mathcal{L}) \,\middle|\, \begin{array}{l} C \text{ nodal} \\ \mathcal{L}/C \text{ line bdle} \end{array} \right\}$$

→ $S_{\mathcal{L}(\alpha)}^{-1}(\overline{e}) = DRL_A \subseteq M^\diamond_{g,A}$

→ In particular: $S_{\mathcal{L}(\alpha)}$ maps $\overline{M}^\diamond_{g,A} \setminus M^\diamond_{g,A}$ away from $\overline{e}$

(stability ⇒ ∄ twist of $(\hat{C},\mathcal{L}(\alpha))$ to multideg. 0)

$$\Rightarrow \log DR_g(A) = \widehat{DR}_g(A) = S_{\mathcal{L}(\alpha)}^*([\overline{e}])$$

Thm (Bae-Holmes-Pandh.-S.-Schwarz+ε)

$$[\overline{e}] = \left[\exp(\hat{\eta}) \cdot \Phi(\mathcal{F}_P) \right]_g$$

$$\Big\downarrow S^*_{\mathcal{L}(\alpha)}$$

$$\log DR_g(A) = \left[\exp(\eta + \Phi(\varphi_L)) \cdot \Phi(\varphi_P) \right]_g$$

Thank you for your attention!

Pixton's formula as a piecewise polynomial

Montag, 22. August 2022 09:54

Thm (HMPPS)

$\exists\, F_A \in sPP(\Sigma_{\overline{\mathcal{M}}_{g,n}}) : DR_g(A) = \left[\exp\left(\sum_{i=1}^{n} \frac{a_i^2}{2}\right) \cdot \Phi(F_A)\right]_g \in CH^g(\overline{\mathcal{M}}_{g,n}).$

Formula for F_A:

Given Γ and $r \in \mathbb{Z}_{>0}$, an admissible weighting w mod r on Γ is a map $w : H(\Gamma) \longrightarrow \{0, 1, \ldots, r-1\}$ such that

- $w(h) + w(h') \equiv 0 \mod r$ for $e = (h, h')$ edge
- $w(l_i) \equiv a_i \mod r$ for l_i: leg assoc. to marking i
- $\sum_{\substack{h \in H(\Gamma) \\ \text{at } v}} w(h) \equiv 0 \mod r$ for v vertex.

Then

$$F_A^r\Big|_{\sigma_\Gamma} = r^{-h^1(\Gamma)} \cdot \sum_{\substack{w \text{ ad.wt.} \\ \text{on } \Gamma}} \prod_{\substack{e=(h,h') \\ \in E(\Gamma)}} \exp\left(\frac{w(h)\,w(h')}{2} \cdot l_e\right)$$

cone in $\Sigma_{\overline{\mathcal{M}}_{g,n}}$ associated to Γ

length of edge e $\hat{=}$ coordinate funct. on σ_Γ

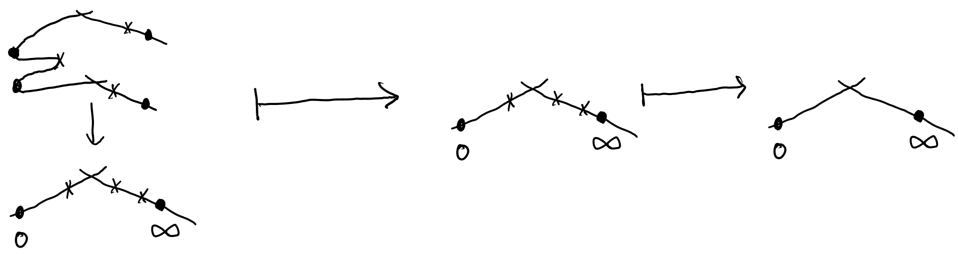
$$\rightsquigarrow F_A = F_A^r\Big|_{r=0} \qquad \#$$

polynomial in r for $r \gg 0$

Double Hurwitz numbers from LogDR

Donnerstag, 25. August 2022 14:43

$$\begin{tikzcd}
 & & \overline{\mathcal{M}}_{g,n} & & \\
 & \widehat{\mathcal{M}} \arrow[ur, "\text{log-blowup}"] \arrow[rrrd, bend left, "\widetilde{br}"] & & & \\
\overline{\mathcal{M}}_{g,A}(\mathbb{P}^1,0,\infty)^\sim \arrow[ur, "F"] \arrow[rr, "br"] & & {[LM_b/S_b]} \arrow[rr, "G"] & & J
\end{tikzcd}$$



$$h_{g,A} = \int_{[\overline{\mathcal{M}}_{g,A}(--)^\sim]^{vir}} br^*[pt]$$

$$= \int_{[\overline{\mathcal{M}}_{g,A}(--)^\sim]^{vir}} br^* G^* \underbrace{\overset{\text{b components}}{[\ \cdots\ ]}}_{P_b \subseteq J}$$

$$= \int_{\widehat{\mathcal{M}}} \underbrace{\left(F_*[\overline{\mathcal{M}}_{g,A}(--)^\sim]^{vir}\right)}_{= \log DR_A} \cdot \widetilde{br}[P_b]$$