1) Introduction and Motivation

Montag, 20. April 2020 14:36

Goal of lecture: Introduce moduli spaces of alg. curves
Goal of mod. spaces: Classify algebraic curves up to isomorph.

Example (Linear Algebra)

$$\left\{ \begin{array}{c}\text{fin. dim'l vector spaces} \\ \text{over field } K\end{array} \right\} / \text{iso} \xrightarrow{\sim} \{0,1,2,\ldots\} = \mathbb{Z}_{\geq 0}$$

$$[V] \longmapsto \dim V$$

⇝ great example of classification

Q What about algebraic geometry?
⇝ classify algebraic $\underbrace{\text{varieties}}_{\substack{\text{reduced, separated schemes of} \\ \text{fin. type over field } K = \bar{K}}}$ up to isom.

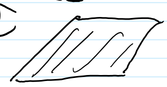
dim 0 : finite unions of points Spec K

dim 1 ⇝ this course

dim > 1 ⇝ more complicated

Def A curve is a variety of pure dimension 1 over K.

Focus on case $K = \mathbb{C}$.
⇝ V complex variety
$\Rightarrow V(\mathbb{C})$ has complex topology
e.g. $V = \mathbb{A}^1 \rightsquigarrow V(\mathbb{C}) = \mathbb{C}$



Def

$$M = \{ C \text{ smooth, irreducible, complex projective curve} \} / \text{iso}$$

→ smooth: easier than singular curves
⇝ later: useful to allow some singularities

→ irreducible: C smooth: C irreducible $\Leftrightarrow$ C connected

∃ point in two irred. comp.
⇝ singular point

→ complex: complex topology

→ projective: Thm C' smooth, irred., complex, not-nec. proj. curve
$\Rightarrow \exists\, C$ " " " proj. " :

$$C' \xhookrightarrow{\text{open}} C$$

$$C \setminus C' = \{p_1, \ldots, p_n\} \subset C$$

$\Rightarrow$ easier to classify $(C, p_1, \ldots, p_n)$

Examples of elements in M

- projective line $\mathbb{P}^1$

- $V(f) \subseteq \mathbb{P}^2$ vanishing set of a homog. polynomial
- function fields over $\mathbb{C}$ $\left[[C] \in \mathcal{M} \rightsquigarrow \{\text{rat'l fcts on } C\} \to \text{fund. fields over } \mathbb{C} \right]$
- elliptic curves $(E, p) = \mathbb{C}/\Lambda$ ← lattice

Ex1 (Projective line)

$$\mathbb{P}^1 = U_0 \cup U_1, \quad U_0 = \{[X:Y] : X \neq 0\} = \mathbb{P}^1 \setminus \{[0:1]\} \cong \mathbb{A}^1 \qquad [1:Y] \leftarrow Y$$

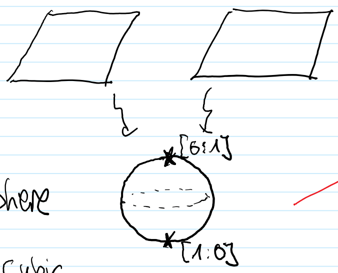
$$U_1 = \{[X:Y] : Y \neq 0\} = \mathbb{P}^1 \setminus \{[1:0]\} \cong \mathbb{A}^1$$

$$\mathbb{P}^1(\mathbb{C}) = \underset{\mathbb{C}}{U_0(\mathbb{C})} \cup \underset{\mathbb{C}}{U_1(\mathbb{C})}$$

$$U_0 \cap U_1 = \mathbb{A}^1 \setminus pt = \mathbb{P}^1 \setminus \{[0:1], [1:0]\}$$

$$U_0 \cap U_1(\mathbb{C}) = \mathbb{C} \setminus pt$$

Sphere



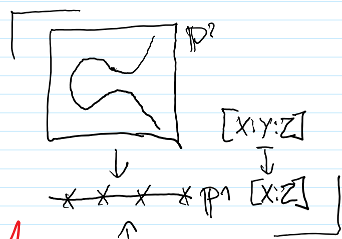
$\to g = 0$

Ex2 (Elliptic curves)

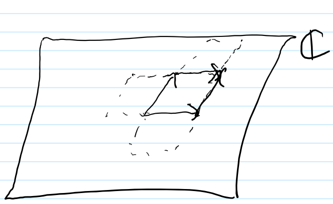
Consider the family E_t of cubic curves in $\mathbb{P}^2$

$$E_t = \{[X:Y:Z] \mid Y^2 Z + X(X-Z)(X-tZ) = 0\} \subset \mathbb{P}^2$$

One checks: For $t \neq 0, 1$, E_t is smooth $\rightsquigarrow [E_t] \in \mathcal{M}$ $\quad t \in \mathbb{C}$

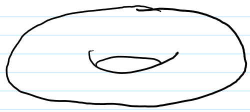


ramified over $[0:1], [1:1], [t:1], [1:0]$



$E_t(\mathbb{C})$

$\rightsquigarrow \mathbb{C}/\Lambda$

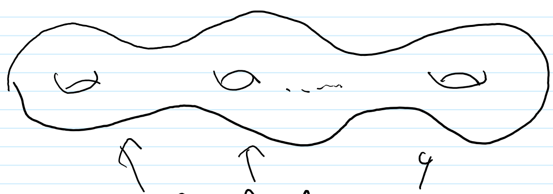


torus $S^1 \times S^1$

$\to g = 1$

More general $[C] \in \mathcal{M} \rightsquigarrow C(\mathbb{C})$ are an oriented, compact real surface

("Surface of a donut w/ g holes")



g holes $\rightsquigarrow$ genus of the surface (or of the curve)

Def $[C] \in \mathcal{M}$

$$g(C) = \dim H^0(C, \Omega^1_C) \qquad \text{"geometric genus"}$$

↑ cotangent line bundle

Exercise Show $g(\mathbb{P}^1) = 0$, $g(E_t) = 1$. $\quad (t \in \mathbb{C} \setminus \{0,1\} = U)$.

There exists map

$$\begin{aligned} \mathcal{M} &\longrightarrow \{0,1,2,\dots\} \\ [C] &\longmapsto g(C) \end{aligned} \Bigg\} \text{ Define } \mathcal{M}_g \text{ as preim. of } g \in \mathbb{Z}_{\geq 0}$$

<u>Fact 1</u> $\mathcal{M}_0 = \{[\mathbb{P}^1]\}$

<u>Fact 2</u> $\mathcal{M}_1 = \{[E_t] \mid t \in U = \mathbb{C}\setminus\{0,1\}\}$

Moreover: $E_{t_1} \cong E_{t_2}$ $(t_1, t_2 \in U)$

iff $t_2 \in \left\{ t_1, \frac{1}{t_1}, 1-t_1, \frac{1}{1-t_1}, \frac{t_1-1}{t_1}, \frac{t_1}{t_1-1} \right\}$ $(\star)$

Define $t_1 \sim t_2$ iff $(\star)$.

This generalizes: $\mathcal{M}_g$ is uncountable for $g \geq 1$.

<u>For E_t over U</u>

$\exists$ action of S_3 (symm. group) on U:

$(12).t = \frac{1}{t}$ $\quad (23).t = 1-t$

$\rightsquigarrow$ orbit of t_1 under S_3 = set $(\star)$ = equiv. classes of $\sim$

$\Rightarrow U/\sim \; = U/S_3$

$\exists$ map $j : U \longrightarrow \mathbb{C} = U/S_3$ "j-invariant"

$$t \longmapsto 2^8 \cdot \frac{(t^2-t+1)^3}{t^2(t-1)^2}$$

s.t. $j(t_1) = j(t_2) \Leftrightarrow t_1 \sim t_2 \Leftrightarrow t_1 \in S_3.t_2$

<u>In fact</u> $U = (\mathbb{A}^1\setminus\{0,1\})(\mathbb{C}) \xrightarrow{j} \mathbb{C} = \mathbb{A}^1(\mathbb{C})$

algebraic

$\Rightarrow \mathcal{M}_1 \cong \mathbb{C}$

In fact, it will be true for all $g \geq 1$:

- $\exists$ algebraic variety $U = U_g$ and family C_t of genus g curves parametrized by $t \in U_g(\mathbb{C})$ such that

$$\mathcal{M}_g = \{[C_t] \mid t \in U_g\}$$

- U_g is smooth, connected variety

- $\exists$ algebraic variety M_g and surjective morphism
  $U_g \longrightarrow M_g \rightsquigarrow$ moduli spaces of curves.
  s.t. two $\overset{t_1, t_2}{\mathbb{C}}$-points of U_g map to same pt. in M_g
  $\Leftrightarrow C_{t_1} \cong C_{t_2}$

$\Rightarrow \mathcal{M}_g = M_g(\mathbb{C})$

Some unexpected aspects:

- For $g \geq 2$, the variety M_g is not smooth

  $\mathrm{Sing}(M_g)^{(\mathbb{C})} = \{[C] \mid \mathrm{Aut}(C) \neq \{id_C\}\}$ $(g \geq 4)$

- For $g \geq 1$, the family C_t $(t \in U_g)$ does not descend to M_g along $U_g \to M_g$

  In other words, we can't find "nice" family $(C'_s)_{s \in M_g}$ s.t. $C'_{[C]} \cong C$

  $[C] \in \mathcal{M}_g = M_g(\mathbb{C})$

  However, such family exists over open locus in M_g of curves w/ $\mathrm{Aut}(C) = \{id_C\}$.

Final problem: M_g is not compact $(g \geq 1)$

$\rightsquigarrow$ Solution: find compactification $\underset{\text{compact}}{\overline{M}_g} \underset{\text{open}}{\supseteq} M_g$

Want: $\overline{M}_g$ is still a moduli space

Exa $(E_t)_{t \in \mathbb{C}}$

$\rightsquigarrow$ smooth for $t \neq 0, 1$

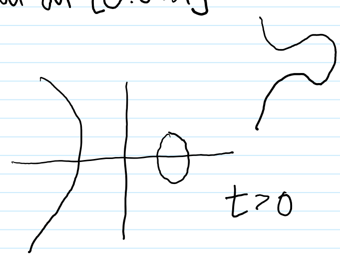
$\rightsquigarrow$ What happens at $t = 0, 1$

$$E_0 = \{[X:Y:Z] : \underbrace{Y^2 Z + X(X-Z)\cdot(X-0\cdot Z)}_{= Y^2 Z + X^2(X-Z)} = 0\} \subseteq \mathbb{P}^2$$

$\nwarrow$ sing. point at $[0:0:1]$

In Chart $\{Z \neq 0\} = \{[x:y:1]\} \cong \mathbb{A}^2$

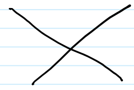
$\rightsquigarrow$ $y^2 + x^2 \underbrace{(x-1)}_{\approx -1 \text{ around } (0,0)} = 0$ Sing. pt $(0,0)$

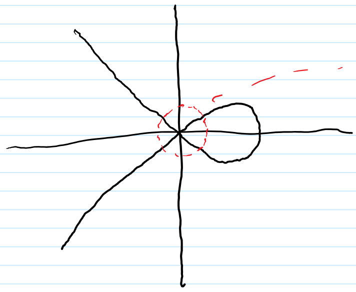


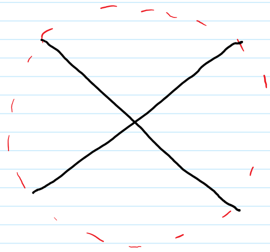
around (0,0) $\rightsquigarrow$ $y^2 - x^2 = 0$

$ \parallel$

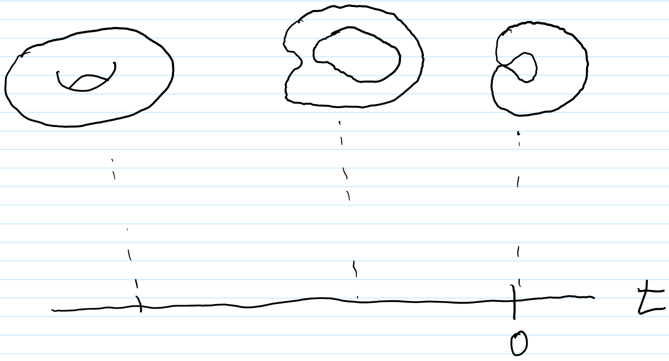
$(y-x)(y+x)$

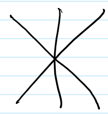






nodal singularity





not node

$\rightsquigarrow$ Define moduli space $\overline{M}_g$ of nodal curves

$\rightsquigarrow$ compact variety, $\overline{M}_g \supset M_g$ open subset