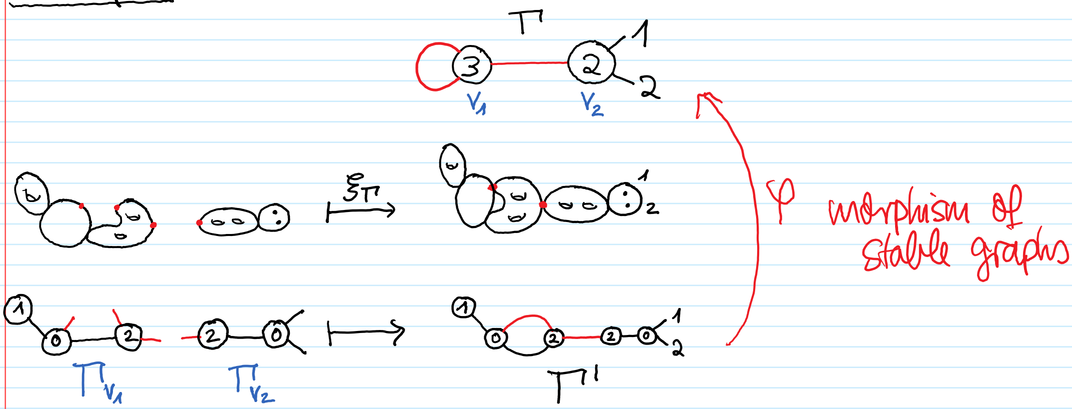
Lecture 10

Montag, 22. Juni 2020 15:53

Example



Exercise Γ stable graph, Γ_v st. graph of genus $g(v)$ w/ $n(v)$ legs for $v \in V(\Gamma)$

$\rightsquigarrow$ define graph Γ' obtained by gluing the Γ_v into vertices of Γ (need bijection: $H_\Gamma(v) \xrightarrow{\sim} \{1,\dots,n(v)\}$ legs of Γ_v)

Show $\left((C_v, p_1^v, \dots, p_{n(v)}^v)\right)_{v \in V(\Gamma)} \in \overline{M}_{\Gamma}$ s.t. Γ_v is stable graph of C_v

$\Rightarrow$ Γ' is stable graph of $\xi_\Gamma((C_v, p_1^v, \dots)_v)$

$\rightsquigarrow$ these are precisely the stable gr. Γ' of curves in $\overline{M}^{\Gamma} = \xi_\Gamma(\overline{M}_\Gamma)$

Def Γ, Γ' stable graphs of genus g w/ n legs.
A **morphism** $\varphi: \Gamma' \to \Gamma$ is defined by two maps

$$\varphi_V: V(\Gamma') \to V(\Gamma), \quad \varphi_H: H(\Gamma) \to H(\Gamma')$$

satisfying the conditions:

a) φ_H is injective

b) φ_H sends edges in Γ to edges in Γ'

$$\{h, h'\} \in E(\Gamma) \Rightarrow \{\varphi_H(h), \varphi_H(h')\} \in E(\Gamma')$$

$\Big\}$ $\varphi_E: E(\Gamma) \to E(\Gamma')$, $\{h,h'\} \mapsto \{\varphi_H(h), \varphi_H(h')\}$.

c) φ_H sends legs of Γ to corresp. legs of Γ'

$$\ell_{\Gamma'}(\varphi_H(h)) = \ell_\Gamma(h) \qquad h \in L(\Gamma)$$

d) φ_V is surjective and compatible w/ φ_H:

$$\varphi_V(v_{T'}(\varphi_H(h))) = v_T(h) \qquad h \in H(T)$$

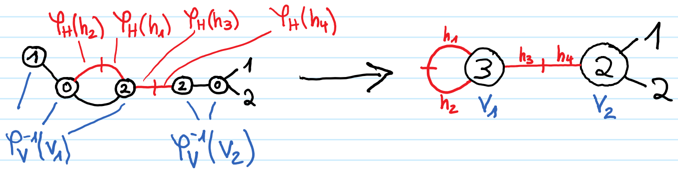
e) Given $v_0 \in V(T)$ the preimage of v_0 under φ_V is a stable graph T'_{v_0} of genus $g(v_0)$ with $n(v_0)$ legs. More precisely, the vertices $V_{v_0} = \varphi_V^{-1}(v_0)$ together w/ half-edges $H_{v_0} = v_{T'}^{-1}(V_{v_0})$ and all edges $\{h, h'\} \in E(T')$ w/ $h, h' \in H_{v_0}$ but

$$\{h, h'\} \neq \{\varphi_H(\tilde{h}), \varphi_H(\tilde{h}')\} \text{ w/ } \{\tilde{h}, \tilde{h}'\} \in E(T)$$

form a stable graph T'_{v_0} of genus $g(v_0)$ w/ $n(v_0)$ legs.

Can obtain T' by gluing T'_{v_0} into vertices v_0 of T

Example



$$T' \longrightarrow T$$
$$\varphi_V : V(T') \twoheadrightarrow V(T)$$
$$H(T') \hookleftarrow H(T) : \varphi_H$$

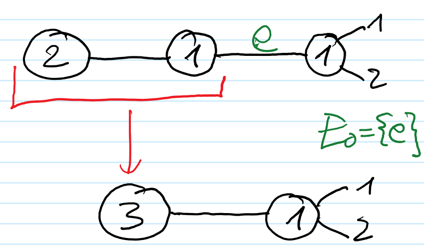
Remarks

a) $\exists\, T' \to T \iff T'$ obtained from T by gluing in T'_{v_0} at vert.

b) Given $T' \xrightarrow{\varphi} T$ $\exists$ a natural gluing morphism

$$\xi_\varphi : \overline{M}_{T'} \longrightarrow \overline{M}_T$$

$$\begin{array}{cc}
\overline{M}_{T'} = \underbrace{\overline{M}_{2,1} \times \overline{M}_{1,2}} \times \underbrace{\overline{M}_{1,3}} & T' \\
\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow & \downarrow \\
\overline{M}_T = \overline{M}_{3,1} \times \overline{M}_{1,3} & T
\end{array}$$



c) In the literature: morphisms $T' \to T$ are sometimes called a T-structure on T', T' is a specializ. of T

T is an edge-contract. of T'

d) Check $\exists$ category with

$\to$ Ob : stable graphs Γ (genus g, n marks.)

$\to$ $\mathrm{Mor}(\Gamma', \Gamma)$: as described above

In particular: $\exists$ composition $\Gamma'' \xrightarrow{\psi} \Gamma' \xrightarrow{\varphi} \Gamma$

$$(\varphi\circ\psi)_V : \quad V(\Gamma'') \xrightarrow{\psi_V} V(\Gamma') \xrightarrow{\varphi_V} V(\Gamma)$$

$$H(\Gamma'') \xleftarrow{\psi_H} H(\Gamma') \xleftarrow{\varphi_H} H(\Gamma) \ : \ (\varphi\circ\psi)_H$$

$\rightsquigarrow$ Check Isom. of stable graphs $\hat{=}$ Isom. in this category.

Exercise

a) Given stable graph Γ' and $E_0 \subset E(\Gamma')$ show

$\exists \Gamma$ stable gr. $\quad \exists\ \Gamma' \longrightarrow \Gamma$

st. $\varphi_E(E(\Gamma)) = E_0 \subseteq E(\Gamma')$

and $\Gamma' \longrightarrow \Gamma$ unique up to isomorph. of Γ.

(contraction of edges in $E(\Gamma') \setminus E_0$)

b) Given $\Gamma' \to \Gamma$ this can be factored

$$\Gamma' = \Gamma_0 \xrightarrow{\varphi_1} \Gamma_1 \xrightarrow{\varphi_2} \Gamma_2 \longrightarrow \cdots \xrightarrow{\varphi_d} \Gamma_d = \Gamma$$

st. each φ_i contracts a single edge of Γ_{i-1}.

Prop Γ stable graph, genus g, n legs. Then a curve $(C, p_1, \ldots, p_n) \in \overline{M}_{g,n}$ lies in $\overline{M}^{\Gamma}$ iff the stable graph Γ' of $(C, p_1, \ldots, p_n)$ admits a morphism $\Gamma' \longrightarrow \Gamma$.

Proof of Prop. Γ' st. gr. of $(C, p_1, \ldots, p_n)$

$(C, p_1, \ldots, p_n) \in \overline{M}^{\Gamma} = \xi_\Gamma(\overline{M}_\Gamma)$

$\Leftrightarrow \Gamma'$ obtained by gluing graphs at vert. of Γ

Rmk a)

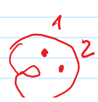
$\Leftrightarrow \Gamma' \xrightarrow{\exists} \Gamma$ $\square$

Quest a) $\quad \overline{M}^{\Gamma} = \bigcup_{\Gamma' \to \Gamma} M^{\Gamma'}$

Cor Γ_1, Γ_2 stable graphs, gen. g, n legs

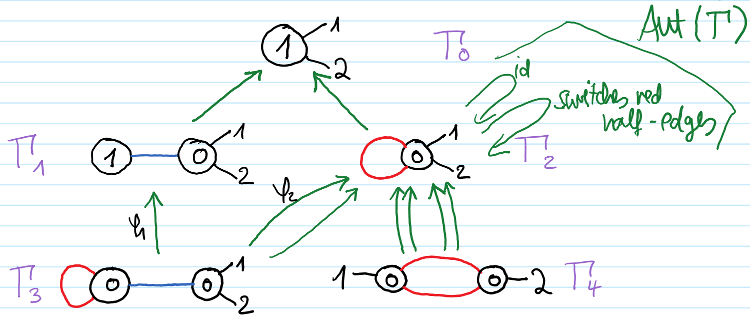
$$\overline{M}^{\Gamma_1} \cap \overline{M}^{\Gamma_2} = \bigcup_{\substack{\Gamma' \\ \exists \Gamma' \to \Gamma_1,\ \exists \Gamma' \to \Gamma_2}} M^{\Gamma'}$$

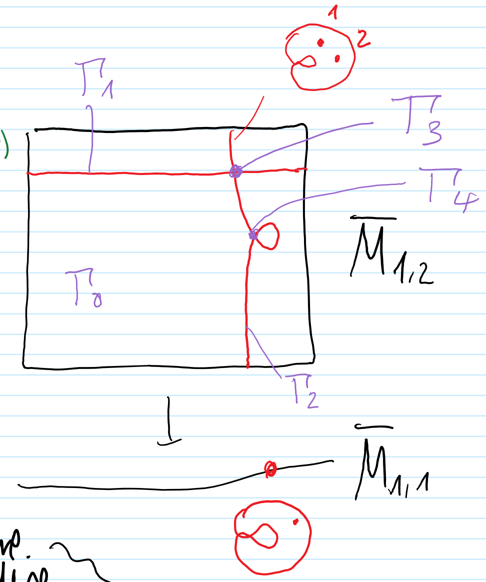
Quest b)



Example $g = 1,\ n = 2$

$\xi_{\Gamma'} : \overline{\mathcal{M}}_{\Gamma'} \to \overline{\mathcal{M}}^{\Gamma'}$ of degree $\#\mathrm{Aut}(\Gamma')$





↳ $(\Gamma_3, \varphi_1, \varphi_2)$ is generic (Γ_1, Γ_2)-structure.
$(\Gamma_3, \varphi_1, \varphi_1)$ is non-gen. (Γ_1, Γ_1)-structure. $\rightsquigarrow$ $\mathcal{G}_{\Gamma_1,\Gamma_2} = \{(\Gamma_3, \varphi_1, \varphi_2)\}$.

More refined question

b') Given Γ_1, Γ_2 st. graphs gen. g, n legs.

$$\begin{tikzcd} \mathcal{F}_{\Gamma_1,\Gamma_2} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_{\Gamma_2} \arrow[d, "\xi_{\Gamma_2}"] \\ \overline{\mathcal{M}}_{\Gamma_1} \arrow[r, "\xi_{\Gamma_1}"] & \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

$\rightsquigarrow$ what is $\mathcal{F}_{\Gamma_1,\Gamma_2}$
(fibre prod. of $\xi_{\Gamma_1}, \xi_{\Gamma_2}$)?

Need definition

Def $\Gamma_1, \Gamma_2, \Gamma'$ st. graphs. A (Γ_1, Γ_2)-structure on Γ' is a tuple

$$(\Gamma', \varphi_1, \varphi_2) = (\Gamma', \varphi_1 : \Gamma' \to \Gamma_1, \varphi_2 : \Gamma' \to \Gamma_2)$$

morph. of stable graphs.

Given second graph Γ'' and (Γ_1, Γ_2) structure $(\Gamma'', \psi_1, \psi_2)$

we say this is isomorphic to $(\Gamma', \varphi_1, \varphi_2)$ if there exists isomorphism $\Gamma' \xrightarrow{\sim} \Gamma''$ fitting in diagram

$$\begin{tikzcd} & \Gamma_1 & \\ \Gamma' \arrow[ur, "\varphi_1"] \arrow[rr, "\sim"] \arrow[dr, "\varphi_2"'] & & \Gamma'' \arrow[ul, "\psi_1"'] \arrow[dl, "\psi_2"] \\ & \Gamma_2 & \end{tikzcd}$$

$\varphi_1 \nearrow \Gamma_1$

$$\xrightarrow{\varphi_2} T_2' \xleftarrow{\psi_2}$$

We say that $(T', \varphi_1, \varphi_2)$ is <u>generic</u> if

$$\begin{tikzcd} & T_1' \\ T' \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & \\ & T_2' \end{tikzcd}$$

$$\rightsquigarrow \begin{tikzcd} & E(T_1') \arrow[dl, "\varphi_{1,E}"'] \\ E(T') & \\ & E(T_2') \arrow[ul, "\varphi_{2,E}"] \end{tikzcd}$$

$$E(T') = \varphi_{1,E}(E(T_1')) \cup \varphi_{2,E}(E(T_2'))$$

$$\mathcal{G}_{T_1',T_2'} = \left\{ (T', \varphi_1, \varphi_2) \;\middle|\; \begin{array}{l} \text{generic} \\ (T_1',T_2')\text{-structure} \end{array} \right\} \Big/ \text{iso}$$

<u>Thm</u> (Graber–Pandharipande ≈ 2005)
Given T_1', T_2' st. graphs, genus g, n legs
the fibre product

$$\begin{tikzcd} \mathcal{F}_{T_1',T_2'} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{T_2'} \arrow[d, "\xi_{T_2'}"] \\ \overline{\mathcal{M}}_{T_1'} \arrow[r, "\xi_{T_1'}"] & \overline{\mathcal{M}}_{g,n} \end{tikzcd} \qquad (\star)$$

$$\begin{tikzcd} \mathcal{F}_{T_1',T_2'} \supseteq \overline{\mathcal{M}}_{T'} \arrow[r, "\xi_{\varphi_2}"] \arrow[d, "\xi_{\varphi_1}"'] & \overline{\mathcal{M}}_{T_2'} \arrow[d, "\xi_{T_2'}"] \\ \overline{\mathcal{M}}_{T_1'} \arrow[r, "\xi_{T_1'}"] & \overline{\mathcal{M}}_{g,n} \end{tikzcd} \qquad (\star\star)$$

is isomorphic to

$$\mathcal{F}_{T_1',T_2'} = \coprod_{(T',\varphi_1,\varphi_2) \in \mathcal{G}_{T_1',T_2'}} \overline{\mathcal{M}}_{T'}$$

The restrict of $(\star)$ to compon. $\overline{\mathcal{M}}_{T'}$ ass to $(T', \varphi_1, \varphi_2)$ is given by $(\star\star)$.

<u>Proof</u> (on level of $\mathbb{C}$-points)
What is a point of $\mathcal{F}_{T_1',T_2'}$?

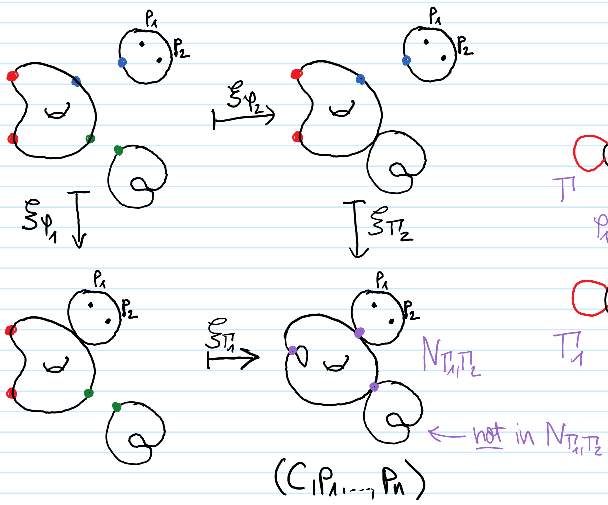
<u>Data</u>

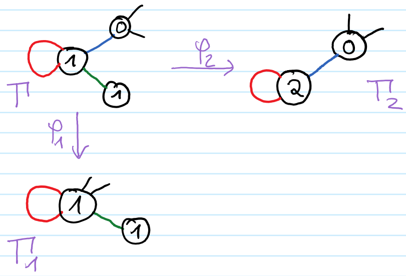
$\rightarrow \left(C_v', (q_h')_{h \in H(T_1')}\right)_{v \in V(T_1')} \in \overline{\mathcal{M}}_{T_1'}$

$\rightarrow \left(C_v'', (q_h'')_{h \in H(T_2')}\right)_{v \in V(T_2')} \in \overline{\mathcal{M}}_{T_2'}$

$\rightarrow$ isomorph. $\xi_{T_1'}(C_v', \ldots) \xrightarrow[\alpha]{\sim} \xi_{T_2'}(C_v'', \ldots) = (C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}$

<u>Normally</u>
$\xi_{T_1'}(C_v', \ldots) \cong \xi_{T_2'}(C_v'', \ldots)$





..... to be continued .....