Lecture 11

Dienstag, 30. Juni 2020 09:18

Thm (Graber–Pandharipande ≈ 2005)
Given Γ_1, Γ_2 st. graphs, genus g, n legs the fibre product

$$\begin{tikzcd} F_{\Gamma_1,\Gamma_2} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{\Gamma_2} \arrow[d, "\xi_{\Gamma_2}"] \\ \overline{\mathcal{M}}_{\Gamma_1} \arrow[r, "\xi_{\Gamma_1}"] & \overline{\mathcal{M}}_{g,n} \end{tikzcd} \qquad (\star)$$

$$\begin{tikzcd} F_{\Gamma_1,\Gamma_2} \supseteq \overline{\mathcal{M}}_{\Gamma'} \arrow[r, "\xi_{\varphi_2}"] \arrow[d, "\xi_{\varphi_1}"] & \overline{\mathcal{M}}_{\Gamma_2} \arrow[d, "\xi_{\Gamma_2}"] \\ \overline{\mathcal{M}}_{\Gamma_1} \arrow[r, "\xi_{\Gamma_1}"] & \overline{\mathcal{M}}_{g,n} \end{tikzcd} \qquad (\star\star)$$

is isomorphic to

$$F_{\Gamma_1,\Gamma_2} = \coprod_{(\Gamma',\varphi_1,\varphi_2)\in \mathcal{G}_{\Gamma_1,\Gamma_2}} \overline{\mathcal{M}}_{\Gamma'}$$

The restrict. of $(\star)$ to compon. $\overline{\mathcal{M}}_{\Gamma'}$ ass. to $(\Gamma',\varphi_1,\varphi_2)$ is given by $(\star\star)$.

Proof (on level of $\mathbb{C}$-points)
What is a point of F_{Γ_1,Γ_2}?

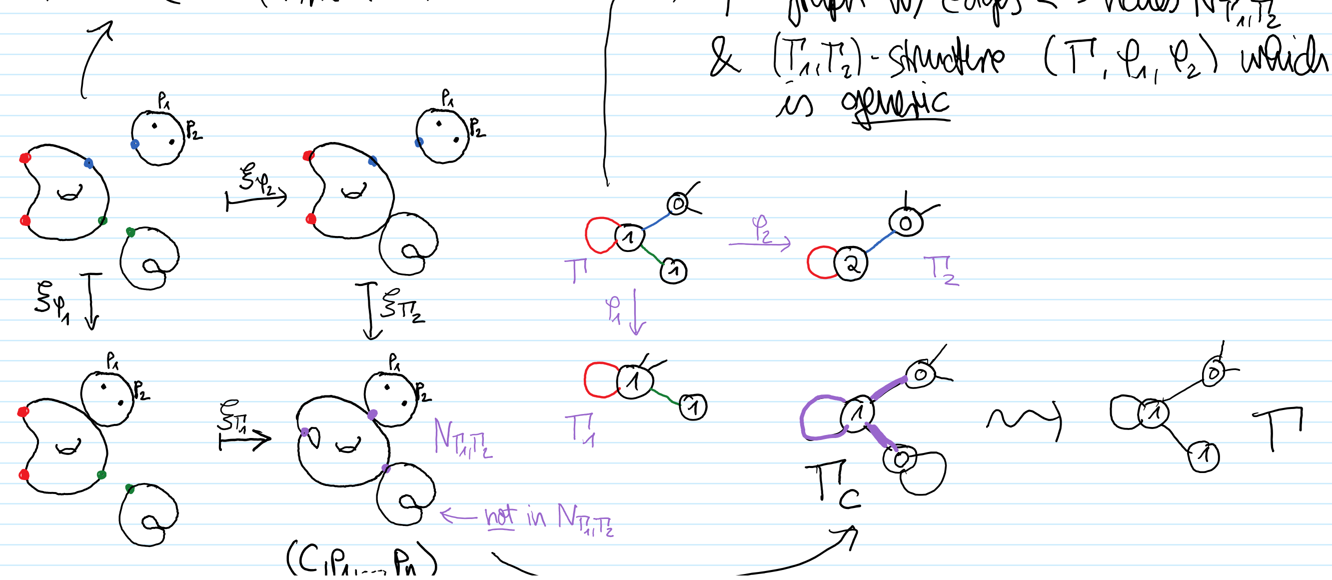
Data

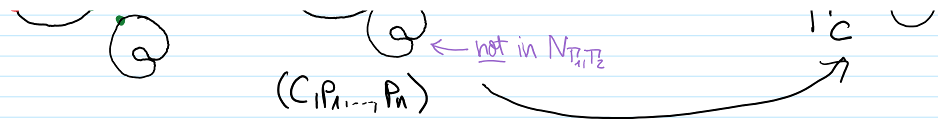
Normally
$\xi_{\Gamma_1}(C_v',\dots) \cong \xi_{\Gamma_2}(C_v'',\dots)$

- $\to \left(C_v', (q_h')_{h\in H(\Gamma_1)}\right)_{v\in V(\Gamma_1)} \in \overline{\mathcal{M}}_{\Gamma_1}$
- $\to \left(C_v'', (q_h'')_{h\in H(\Gamma_2)}\right)_{v\in V(\Gamma_2)} \in \overline{\mathcal{M}}_{\Gamma_2}$
- $\to$ isomorph. $\xi_{\Gamma_1}(C_v',\dots) \xrightarrow[\alpha]{\sim} \xi_{\Gamma_2}(C_v'',\dots) = (C,p_1,\dots,p_n) \in \overline{\mathcal{M}}_{g,n}$

$\overline{\mathcal{M}}_{\Gamma'} \ni \left(\left(C_{\tilde v}, (q_{\tilde h})_{\tilde h\in H(\Gamma')}\right)\right)_{v\in V(\Gamma')}$

Γ' graph w/ edges $\leftrightarrow$ nodes N_{Γ_1,Γ_2} & (Γ_1,Γ_2)-structure $(\Gamma',\varphi_1,\varphi_2)$ which is generic





$$\mathcal{F}_{\Gamma_1,\Gamma_2} \xrightarrow{\sim} \coprod_{(\Gamma, \varphi_1, \varphi_2)} \overline{\mathcal{M}}_{\Gamma}$$

Construct map in opposite direction via diagrams (☆☆)

⇝ Check that these functors are inverse to each other. □

A crash course in intersection theory

⇝ Setting: Smooth, proper complex varieties X

⇝ When discussing int. theory on stacks:
almost everything works like for varieties,
with a few adaptions (mention those below)

[Online reading course "Intersection theory on stacks"
(organized by Reinier Kramer)]

Singular homology & cohomology

X connected, smooth proper variety / $\mathbb{C}$, $\dim_{\mathbb{C}} X = d$

⇝ $(X(\mathbb{C}),$ complex topology$)$ smooth manifold, compact, connected, oriented
$\dim_{\mathbb{R}} X(\mathbb{C}) = 2d$

$$H_*(X) = H_*(X(\mathbb{C}), \mathbb{Q}) \qquad H^*(X) = H^*(X(\mathbb{C}), \mathbb{Q})$$

(Singular) homology (Singular) cohomology

Cap & cup product

$\sigma \in H_k(X)$, $\alpha \in H^\ell(X)$ ⇝ $\sigma \cap \alpha \in H_{k-\ell}(X)$ cap product

For $k = \ell$:

$$\begin{array}{ccc} H_k(X) \otimes H^k(X) & \xrightarrow{\cap} & H_0(X) \overset{X \text{ connected}}{\cong} \mathbb{Q} \cdot [pt] \\ \sigma \otimes \alpha & \longmapsto & \sigma \cap \alpha \end{array}$$

bilinear pairing, nondegenerate $\Rightarrow$ $H^k(X) = H_k(X)^\vee$

The isomorphism

$$H_0(X) \xrightarrow[\sim]{\deg} \mathbb{Q}$$

$$\sum a_i [p_i] \longmapsto \sum a_i$$

is called the degree map.

Cup product

$$\alpha \in H^k(X), \quad \beta \in H^\ell(X) \rightsquigarrow \alpha \cup \beta \in H^{k+\ell}(X)$$

$\rightsquigarrow$ $H^*(X)$ becomes a ring under this product ($\alpha \cdot \beta = \alpha \cup \beta$)

intersection product

Rmk X stack, like $X = \overline{\mathcal{M}}_{g,n} \ni P = (C, p_1, \dots, p_n)$

$\rightsquigarrow$ degree map must be slightly adjusted

$$\deg([P]) = \frac{1}{\# \mathrm{Aut}(P)} = \frac{1}{\# \mathrm{Aut}(C, p_1, \dots, p_n)}$$

$$\deg\left(\sum a_i [P_i]\right) = \sum \frac{a_i}{\# \mathrm{Aut}(P_i)}$$

Exa $X = \mathbb{P}^n$, $n \geq 0$ $\qquad H^k(\mathbb{P}^n) \cong \begin{cases} \mathbb{Q} & , k = 0, 2, 4, \dots, 2n \\ 0 & , \text{otherwise} \end{cases}$

$X = \mathbb{CP}^n$

$\rightsquigarrow$ $H^*(\mathbb{P}^n) \cong \mathbb{Q}[H]/(H^{n+1} = 0)$ $\qquad \mathbb{P}^n = pt \cup \mathbb{C} \cup \mathbb{C}^2 \cup \mathbb{C}^3 \cup \dots \cup \mathbb{C}^n$

$H \in H^2(\mathbb{P}^n) = \mathbb{Q} \cdot H$ $\qquad \dim_{\mathbb{R}}$ 0 2 4 6 ...

see interpretation for this soon

Poincaré duality

Assumpt. on X (connect., smooth, proper) imply:

$$H^k(X) \otimes H^{2d-k}(X) \longrightarrow H^{2d}(X), \quad \alpha \otimes \beta \mapsto \alpha \cup \beta$$

$H^{2d}(X) \cong \mathbb{Q}$

is a perfect pairing. $\rightsquigarrow$ $H^k(X) \cong H^{2d-k}(X)^\vee$

$\cong$ $\leftarrow$ cap prod. pairing

$H_{2d-k}(X)$

With the isom. $H^{2d}(X) \cong \mathbb{Q}$ given by

$$H^{2d}(X) \longrightarrow H_0(X) \cong \mathbb{Q}$$

$$\gamma \longmapsto [X] \cap \gamma \xmapsto{\deg} \deg(\gamma) = \int_X \gamma = \deg([X] \cap \gamma).$$

fund. class $\in H_{2d}(X)$

Isomorphism

$$H^k(X) \xrightarrow{\sim} H_{2d-k}(X), \quad \alpha \mapsto [X] \cap \alpha$$

Exa $X = \mathbb{P}^n$ $\quad (\lambda \cdot H^k) \otimes (\mu \cdot H^{n-k}) \longmapsto (\lambda \cdot \mu) \cdot H^n$

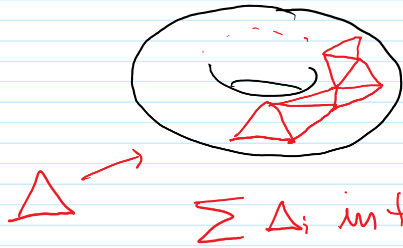
$$\underbrace{(\mathbb{Q} \cdot H^k)}_{H^{2k}(X)} \otimes \underbrace{(\mathbb{Q} \cdot H^{n-k})}_{H^{2n-2k}(X)} \longrightarrow \underbrace{\mathbb{Q} \cdot H^n}_{H^{2n}(X)}$$

Fundamental classes of subvarieties

$X(\mathbb{C})$ connected, closed, oriented manifold

$\rightsquigarrow \exists\, [X] \in H_{2d}(X) \cong \mathbb{Q} \cdot [X]$

↑ fundamental class of X

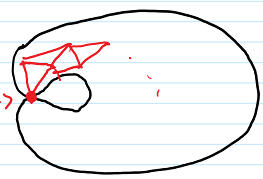


$\sum \Delta_i$ in triangulat. $= [X]$

Can be generalized to (not nec. smooth) subvarieties $Z \subset X$, $\operatorname{codim}_{\mathbb{C}} Z = c$

$$\rightsquigarrow \boxed{[Z] \in H_{2(d-c)}(X) \cong H^{2c}(X)}$$

Sing(X) subcomplex of triangulat.



Generalization $Z \subset X$ subscheme (not subvar.) — e.g. Z nonreduced

$Z = V(x_0^d) \subset \mathbb{P}^1_{[x_0:x_1]}$

To define $[Z]$ let $Z_1, \ldots, Z_r$ be the irred. components of Z^{red}

Z

$m_i = d$

Multiplicity of Z at Z_i:

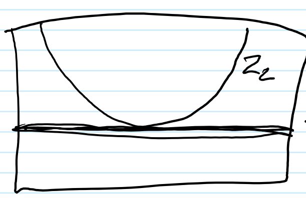
$$m_i = \operatorname{length}_{\mathcal{O}_{Z_i,Z}} \mathcal{O}_{Z_i,Z}$$

scary word from commut. algebra (see references)

Exa $Z = V(f)$ hypersurface

$\rightsquigarrow m_i =$ order of vanishing of f at Z_i

↑ subvar. of X



$\mathbb{A}^2$ $\quad Z_2 \quad Z_1$

Def $[Z] = \sum m_i [Z_i]$

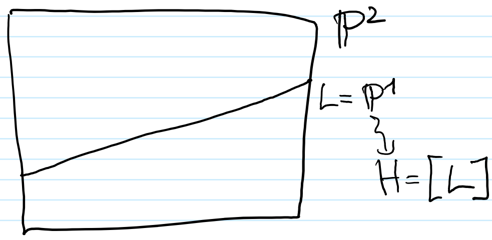
$f = y^5 \cdot (y - x^2)$

$m_1 = 5, \quad m_2 = 1$

Exa $X = \mathbb{P}^n$

$H^2(X) = \mathbb{Q} \cdot H$, $H = [\mathbb{P}^{n-1}]$, $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$ linear subspace codim 1

$H^{2k}(X) = \mathbb{Q} \cdot H^k$, $H^k = [\mathbb{P}^{n-k}]$ ... codim k.



Proper pushforward & flat pullback

X, Y conn., smooth, proper cplx. varieties
$f: X \longrightarrow Y$ morphism. Then we have:

- a pushforward $f_* : H_k(X) \longrightarrow H_k(Y)$
- a pullback $f^* : H^k(Y) \longrightarrow H^k(X)$
  (compatible w/ cup product: $f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$)

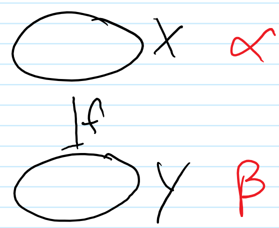
Using Poincaré duality:

$$f_* : H^\ell(X) \longrightarrow H^{\ell + 2(e-d)}(Y) \quad , \quad \dim X = d, \ \dim Y = e$$

Important: Projection formula

$\alpha \in H^*(X)$, $\beta \in H^*(Y)$

$$f_*((f^*\beta) \cup \alpha) = \beta \cup f_*\alpha \in H^*(Y)$$

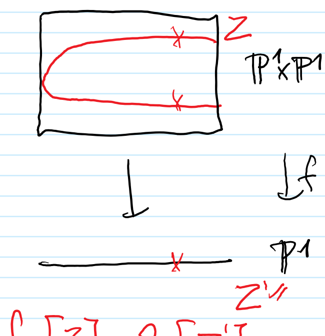


Assume f proper: (X, Y proper $\Rightarrow$ f proper)

$Z \subset X$ subvariety $\rightsquigarrow$ $f(Z) \subset Y$ subvariety, $f(Z) = Z'$

$[Z] \in H^*(X)$

$$f_*[Z] = \begin{cases} \deg(Z/Z') \cdot [Z'] & , \text{ if } \dim Z = \dim Z' \\ 0 & , \text{ otherwise} \end{cases}$$

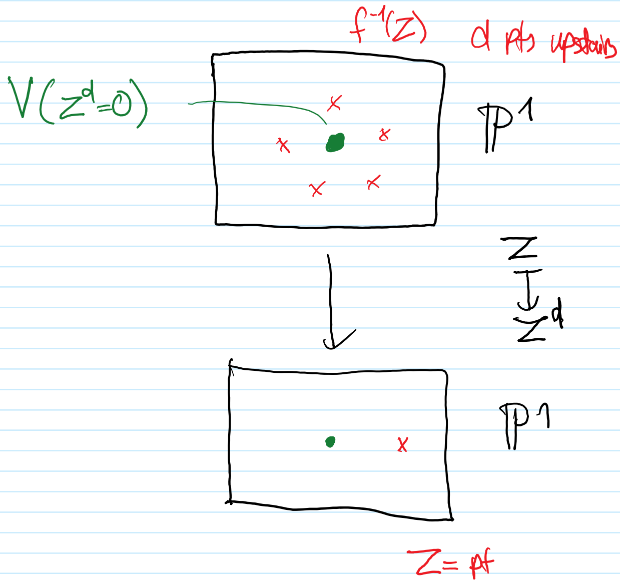


$\deg(Z/Z') := [\mathbb{C}(Z) : \mathbb{C}(Z')]$ degree of field extension

$= \# f^{-1}(z')$ for $z' \in Z'$ general point.

Assume f flat map

$f^{-1}(Z)$ d pts upstairs



Assume f flat map

$Z \subset Y$ subvariety

$$f^*[Z] = [f^{-1}(Z)]$$

↑ scheme-theor. preimage

$$\begin{tikzcd} f^{-1}(Z) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Z \arrow[r, "\text{incl. of } Z"'] & Y \end{tikzcd}$$

Prop Assume we have fibre diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

g flat
f proper

Then g' flat, f' proper and for $\alpha \in H^*(X)$

$$g^* f_* \alpha = (f')_* (g')^* \alpha \in H^*(Y')$$

Exa $X = \mathbb{P}^n$, Given $F \in \mathbb{C}[X_0, \ldots, X_n]_d$ nonzero homog. deg. d polynom.

$\rightsquigarrow S = V(F) \subset \mathbb{P}^n$ hypersurf. cut out by F

$\rightsquigarrow [S] = \,? \in H^2(\mathbb{P}^n) = \mathbb{Q} \cdot H$

For this, consider space $\mathbb{P}^N = \mathbb{P}(\mathbb{C}[X_0, \ldots, X_n]_d)$

smooth, proper connect. variety $\to$ $\mathcal{H} = \{([F], p) \in \mathbb{P}^N \times \mathbb{P}^n \mid F(p) = 0\} \xrightarrow{\pi_2} \mathbb{P}^n$ ($\mathbb{P}^{N-1}$-bundle, proper)

$\downarrow \pi_1 \to$ flat (all fibres have dim $= n-1$)

$\mathbb{P}^N \ni [F]$

$$(\pi_1)^{-1}([F]) = V(F) \rightsquigarrow \pi_1^*[F] = [V(F)]$$

$[F] = [X_0^d] \in H_0(\mathbb{P}^N)$

$$(\pi_2)_* \pi_1^*[F] = [V(F)] \in H^2(\mathbb{P}^n)$$

$$\|$$

$$(\pi_2)_* \pi_1^*[X_0^d] = [V(X_0^d)] = d \cdot [V(X_0)] = d \cdot [\mathbb{P}^{n-1}] = d \cdot H.$$