Lecture 12

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Useful stuff I forgot last time:

$\to$ $\dim_{\mathbb{C}} X = d \Rightarrow H_k(X) = 0,\ H^k(X) = 0$ for $k<0$ or $k>2d$

$\to$ $[X]$ neutral element for $\cup$: $[X] \cup \alpha = \alpha \quad \forall \alpha \in H^*(X)$

$\to$ Functoriality of pullback and pushforward:

Given $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have

$$g_*(f_* \alpha) = (g \circ f)_* \alpha \quad , \alpha \in H^*(X)$$

$$f^*(g^* \beta) = (g \circ f)^* \beta \quad , \beta \in H^*(Z)$$

Chern classes of line bundles

$\mathcal{L}$ line bundle on $X \rightsquigarrow c_1(\mathcal{L}) \in H^2(X)$ (works in more generality)

$$\left\{ \begin{matrix} \text{line bundles} \\ \mathcal{L} \text{ on } X \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{divisor classes} \\ D = \sum a_i D_i \text{ on } X \end{matrix} \right\} \longrightarrow H^2(X)$$

$$\mathcal{L} = \mathcal{O}(D) \longleftarrow D = \sum a_i D_i \longmapsto \sum a_i [D_i]$$

1^{st} Chern class is just composition $\longrightarrow \longrightarrow$.

Easy exercise

$$c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2)$$

$$c_1(\mathcal{L}^\vee) = -c_1(\mathcal{L}).$$

Chern classes of vector bundles

$\mathcal{V}$ vector bundle on $X \rightsquigarrow$ can define $c_i(\mathcal{V}) \in H^{2i}(X)$, $i = 1, \ldots, r$

We only need special case: $\mathcal{V} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$, $i = r$

$$c_r(\mathcal{V}) = c_{top}(\mathcal{V}) = c_1(\mathcal{L}_1) \cup c_1(\mathcal{L}_2) \cup \cdots \cup c_1(\mathcal{L}_r) \in H^{2r}(X)$$

Excess intersection formula

Idea What is $[Z_1] \cup [Z_2]$?

Def A morphism f is called a <u>local complete intersection (l.c.i.)</u>

if it can be written as a composition

$$f = \begin{pmatrix} \text{smooth} \\ \text{morphism} \end{pmatrix} \circ \begin{pmatrix} \text{regular} \\ \text{embedding} \end{pmatrix} \qquad \left(\begin{tikzcd} & X \times Y \arrow[dr, "\pi_Y"] & \\ X \arrow[ur, "\mathrm{id} \times f"] & & Y \end{tikzcd} \right)$$

$\rightsquigarrow f: X \to Y$ w/ X smooth $\rightsquigarrow$ condition automatic

Prop Assume we have a fibre diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

w/ X, Y, Y', X' connected, smooth, proper

does not actually need to be connected

with g, g' being lci morphisms of codimension d, d' and f proper.
For $\alpha \in H^*(X)$ we have

$$g^* f_* \alpha = (f')_* \left(c_{top}(E) \cup (g')^* \alpha \right) \in H^*(Y)$$

where E on X' is the rank $(d-d')$-bundle given as the quotient

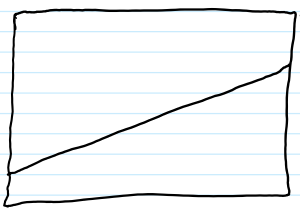
$$E = (f')^* \mathcal{N}_{Y'/Y} / \mathcal{N}_{X'/X}.$$

tangent maps are injective

Concern. def. of E: If f, g are unramified, then definit above makes sense, with normal bundles:

$$\mathcal{N}^\vee_{Y'/Y} = g^*\Omega^1_Y / \Omega^1_{Y'} \quad , \quad \mathcal{N}^\vee_{X'/X} = (g')^*\Omega^1_X / \Omega^1_{X'}$$

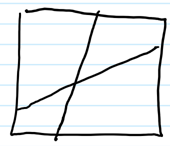
Example



$Y = \mathbb{P}^2$

$Y' = X = \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "id"] \arrow[d, "id"'] & \mathbb{P}^1 \arrow[d, hook, "i"] \\ \mathbb{P}^1 \arrow[r, hook, "i"] & \mathbb{P}^2 \end{tikzcd}$$



$H \in H^2(\mathbb{P}^2)$

$$\underset{H^2(\mathbb{P}^1)}{[pt]} \overset{?}{=} i^* \underbrace{i_*[\mathbb{P}^1]}_{H} = (id)_* \underbrace{\left(c_{top}(E) \cup \underbrace{(id)^*[\mathbb{P}^1]}_{[\mathbb{P}^1]} \right)}_{c_{top}(E)} = c_{top}(E) = c_{top}(\mathcal{O}(1)) = [pt]$$

can use Euler exact sequence on $\mathbb{P}^2$

$$E = id^* \mathcal{N}_{\mathbb{P}^1/\mathbb{P}^2} / \underbrace{\mathcal{N}_{id}}_{=0} = \mathcal{N}_{\mathbb{P}^1/\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}([\mathbb{P}^1])\big|_{\mathbb{P}^1} = \mathcal{O}(1)|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(1)$$

$$\mathcal{N}_{D/X} = \mathcal{O}_X(D)\big|_D$$

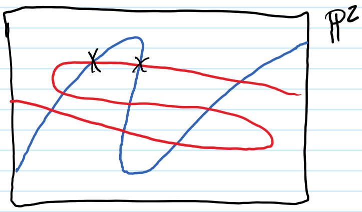
effect. Cartier div on X

Want $\underbrace{[\mathbb{P}^1]\cup[\mathbb{P}^1]}_{\in H^4(\mathbb{P}^2)} = \left(i_*[\mathbb{P}^1]\right)\cup\left(i_*[\mathbb{P}^1]\right)$

$$\overset{\text{Proj. formula}}{=} i_*\left(i^*(i_*[\mathbb{P}^1])\cup[\mathbb{P}^1]\right)$$

$$= i_*\big(\underbrace{[pt]}_{\in H^2(\mathbb{P}^1)}\big) = [pt] \in H^4(\mathbb{P}^2).$$

Exercises

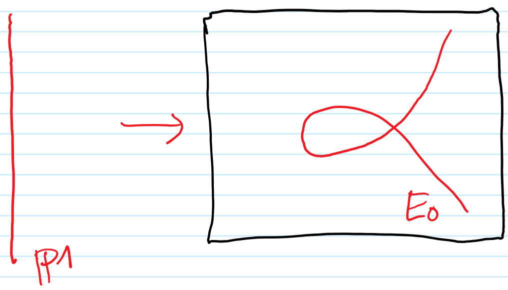


$\rightsquigarrow$ $Z_1 \cap Z_2$ transversally

$\rightsquigarrow [Z_1]\cup[Z_2] = [Z_1\cap Z_2]$

Can use this to show:
$C_1, C_2 \subset \mathbb{P}^2$ curves of degrees d, e intersecting transversally
$\Rightarrow$ intersect in precisely $d\cdot e$ points.

Exercise



What is

$$\underbrace{\left(i_*[\mathbb{P}^1]\right)}_{[E_0]} \cup \left(i_*[\mathbb{P}^1]\right) \; ?$$

$[E_0] \overset{\text{Example}}{=} 3\cdot H$

$\left(i_*[\mathbb{P}^1]\right)\cup\left(i_*[\mathbb{P}^1]\right) \overset{!}{=} 9\cdot H^2 = 9\cdot[pt]$

Intersection theory on $\overline{\mathcal{M}}_{g,n}$

$\rightsquigarrow$ How to define interesting classes in $H^*(\overline{\mathcal{M}}_{g,n})$?

- $\rightsquigarrow$ use forgetful map $\pi: \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$
- $\rightsquigarrow$ use the $\overline{\mathcal{M}}_\Gamma \xrightarrow{\xi_\Gamma} \overline{\mathcal{M}}^\Gamma \subset \overline{\mathcal{M}}_{g,n}$, e.g. $[\overline{\mathcal{M}}^\Gamma] \in H^*(\overline{\mathcal{M}}_{g,n})$
- $\rightsquigarrow$ use cup product.
- $\rightsquigarrow$ $\underset{\text{natural}}{\mathcal{L}}$ line bundle on $\overline{\mathcal{M}}_{g,n} \rightsquigarrow c_1(\mathcal{L}) \in H^2(\overline{\mathcal{M}}_{g,n})$

Def (The tautological ring)
The tautological rings $(RH^*(\overline{\mathcal{M}}_{g,n}))_{g,n}$ are the smallest system of $\mathbb{Q}$-subalgebras of $(H^*(\overline{\mathcal{M}}_{g,n}))_{g,n}$ which contain $[\overline{\mathcal{M}}_{g,n}] \in H^0(\overline{\mathcal{M}}_{g,n})$ and are closed under pushforward by all gluing morphisms

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \longrightarrow \overline{\mathcal{M}}_{g,n}$$

and forgetful maps*

$$\pi : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}.$$

* allow all forg. maps of markings i:
$\pi_i(C, p_1, \dots, p_{n+1})$
$= (C, p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_{n+1})$

Remarks

$\to$ "$\mathbb{Q}$-subalgebra": closed under addition, mult. w/ elem. of $\mathbb{Q}$, cup product.

$\to$ "closed under pushforw. by ξ_Γ"

$$\begin{tikzcd} \overline{\mathcal{M}}_\Gamma = \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)} \arrow[d, "\xi_\Gamma"'] \arrow[r, "\pi_v"] & \overline{\mathcal{M}}_{g(v),n(v)} \\ \overline{\mathcal{M}}_{g,n} & \end{tikzcd}$$

$$(\alpha_v \in RH^*(\overline{\mathcal{M}}_{g(v),n(v)}))_v$$
$$\rightsquigarrow \quad \alpha = \prod_{v \in V(\Gamma)} (\pi_v)^* \alpha_v \in H^*(\overline{\mathcal{M}}_\Gamma)$$

Require $(\xi_\Gamma)_*(\alpha) \in RH^*(\overline{\mathcal{M}}_{g,n})$.

$\to$ $\alpha \in RH^*(\overline{\mathcal{M}}_{g,n})$ are called <u>tautological classes</u>.

$\to$ $\overline{\mathcal{M}}_\Gamma \xrightarrow[\deg = \#\mathrm{Aut}(\Gamma)]{\xi_\Gamma} \overline{\mathcal{M}}^\Gamma \subseteq \overline{\mathcal{M}}_{g,n}$

$$\Rightarrow [\overline{\mathcal{M}}^\Gamma] = \frac{1}{\#\mathrm{Aut}(\Gamma)} (\xi_\Gamma)_* [\overline{\mathcal{M}}_\Gamma] \in RH^*(\overline{\mathcal{M}}_{g,n})$$

defin. of proper pushf.

there are lots of taut. classes!

Goal for rest of the lectures

Understand better what taut. classes are.
$\rightsquigarrow$ give a finite set of generators as $\mathbb{Q}$-vector space.

Need some more ingredients to state this

$$\begin{array}{c} \overline{\mathcal{C}}_{g,n} \cong \overline{\mathcal{M}}_{g,n+1} \\ \pi \downarrow \;\; \uparrow p_i \\ \overline{\mathcal{M}}_{g,n} \end{array}$$

Ω^1_π = sheaf of relative differentials on $\overline{\mathcal{C}}_{g,n} \xrightarrow{\pi} \overline{\mathcal{M}}_{g,n}$

Def For $i = 1, \ldots, n$, the <u>ith cotangent line bundle</u> $\mathbb{L}_i$ on $\overline{\mathcal{M}}_{g,n}$ is defined as

$$\mathbb{L}_i = p_i^* \Omega^1_\pi .$$

We define the <u>ith ψ-class</u> ψ_i as

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}).$$

$(C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}$

$\pi^{-1}((C, p_1, \ldots, p_n)) \cong C$

given $q \in C$ smooth point.

$\rightsquigarrow \Omega^1_\pi |_q = \Omega^1_q \, \pi^{-1}((C, p_1, \ldots, p_n))$

$= \Omega^1_q C = T_q^* C$

$q = p_i$

$\rightsquigarrow \mathbb{L}_i |_{(C, p_1, \ldots, p_n)} = \Omega^1_\pi |_{q = p_i}$

$= T_{p_i}^* C .$

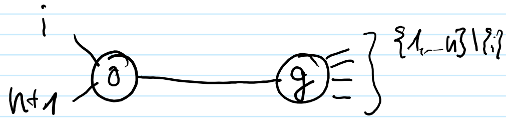
fibre of line bdle $\mathbb{L}_i$ at $(C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}$ is $T_{p_i}^* C$

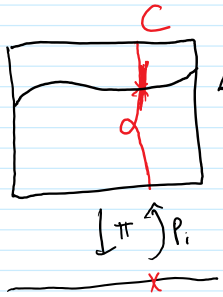
$\rightsquigarrow$ a priori, there is no reason why ψ_i should be tautol.

$\rightsquigarrow$ turns out, it is.

Prop $\psi_i \in RH^*(\overline{\mathcal{M}}_{g,n})$.

Proof Recall: $p_i : \overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{C}}_{g,n} = \overline{\mathcal{M}}_{g,n+1}$ is an example of a gluing morphism.

$p_i = \xi_{\Gamma_i}$, $\Gamma_i =$ 



$\Delta_i = p_i(\overline{\mathcal{M}}_{g,n})$

$\rightsquigarrow [\Delta_i] = [\overline{\mathcal{M}}^{\Gamma_i}]$ is tautological.

<u>Claim</u> $\psi_i = -\pi_*\left(\underbrace{[\Delta_i] \cup [\Delta_i]}_{\in RH^*(\overline{\mathcal{M}}_{g,n+1})}\right)$

<u>Pf of Claim</u> (Claim finishes pf.)

$$[\Delta_i] \cup [\Delta_i] = p_{i*}[\overline{\mathcal{M}}_{g,n}] \cup p_{i*}[\overline{\mathcal{M}}_{g,n}]$$

$$= p_{i*}\Big(\underbrace{p_i^*\big(p_{i*}[\overline{\mathcal{M}}_{g,n}]\big)}_{\text{need this}} \cup [\overline{\mathcal{M}}_{g,n}] \Big)$$

p_i proper, lci, closed embedding (section of π)

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n} \arrow[r, "id"] \arrow[d, "id"'] & \overline{\mathcal{M}}_{g,n} \arrow[d, "p_i"] \\ \overline{\mathcal{M}}_{g,n} \arrow[r, "p_i"] & \overline{\mathcal{M}}_{g,n+1} \end{tikzcd}$$

$\rightsquigarrow$ can see

excess int. bundle

$$p_i^*\big(p_{i*}[\overline{\mathcal{M}}_{g,n}]\big) = c_1(E)$$

$$= c_1\big(\underbrace{\mathcal{N}_{\overline{\mathcal{M}}_{g,n}/\overline{\mathcal{M}}_{g,n+1}}}\big)$$

$$\mathcal{N}^\vee_{\overline{\mathcal{M}}_{g,n}/\overline{\mathcal{M}}_{g,n+1}} = p_i^*\big(\Omega^1_{\overline{\mathcal{M}}_{g,n+1}}\big)/\Omega^1_{\overline{\mathcal{M}}_{g,n}}$$

$$= p_i^*\,\Omega^1_\pi = \mathbb{L}_i$$

$$\Rightarrow \mathcal{N}_{\overline{\mathcal{M}}_{g,n}/\overline{\mathcal{M}}_{g,n+1}} = \mathbb{L}_i^\vee \Rightarrow c_1(\mathbb{L}_i^\vee) = -c_1(\mathbb{L}_i) = -\psi_i$$

$$\Rightarrow [\Delta_i] \cup [\Delta_i] = -\,p_{i*}\psi_i$$

$$\Rightarrow \pi_*\big([\Delta_i]\cup[\Delta_i]\big) = -\,(\pi_* p_{i*})\,\psi_i$$

$$= -\,(\underbrace{\pi\circ p_i}_{id_{\overline{\mathcal{M}}_{g,n}}})_*\,\psi_i = -\psi_i \qquad \square$$

$\rightsquigarrow$ ψ_i are tautological

<u>Def</u> (K-classes)

$$\begin{tikzcd} \overline{\mathcal{C}}_{g,n} = \overline{\mathcal{M}}_{g,n+1} \arrow[d, "\pi"] & \psi_{n+1} \\ \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

Given $a \in \mathbb{Z}_{\geq 0}$, define

$$\kappa_a = \pi_*\big(\underbrace{\psi_{n+1}^{a+1}}_{\in H^{2(a+1)}(\overline{\mathcal{M}}_{g,n+1})}\big) \in RH^{2a}(\overline{\mathcal{M}}_{g,n})$$

taut. class

<u>Next time</u> use ψ, κ, ξ_Γ to write down generat. of $RH^*(\overline{\mathcal{M}}_{g,n})$.