Lecture 13

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Def Γ stable graph. A decoration on Γ is a class $\alpha \in H^*(\overline{M}_\Gamma)$, which is a prod. of κ and ψ-classes on factors $\overline{M}_{g(v),n(v)}$ of $\overline{M}_{\Gamma}$.

$$\begin{tikzcd} \overline{M}_{g,n} \\ \overline{M}_{\Gamma} \arrow[u, "\xi_\Gamma"] \arrow[d, "\pi_v"] \\ \overline{M}_{g(v),n(v)} \end{tikzcd}$$

$$\alpha = \prod_{v \in V(\Gamma)} \pi_v^*(\alpha_v) \quad , \quad \alpha_v = \kappa_1^{e_{v,1}} \cdots \kappa_m^{e_{v,m}} \cdot \psi_1^{f_{v,1}} \cdots \psi_{n(v)}^{f_{n(v),1}} \quad \in RH^*(\overline{M}_{g(v),n(v)})$$

Given Γ, α, we define a decorated stratum class $[\Gamma, \alpha]$ as

$$[\Gamma, \alpha] = (\xi_\Gamma)_* \alpha \quad \in \quad RH^*(\overline{M}_{g,n})$$

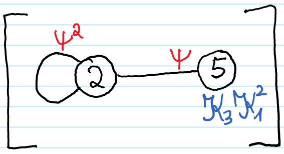
Rmk

→ only fin. many nonzero decorat. α on given Γ, bc. $\overline{M}_{g(v),n(v)}$ fin.dim'l

$H^k(\overline{M}_{g(v),n(v)}) = 0$, $k > 2(3g(v)-3+n(v))$

→ If $\alpha \in H^{2d}(\overline{M}_\Gamma) \rightsquigarrow [\Gamma, \alpha] \in RH^{2(d+e)}(\overline{M}_{g,n})$

since ξ_Γ is of $^{\mathbb{C}}$codim $e := \#E(\Gamma)$



$$= (\xi_\Gamma)_* \Big(\underbrace{\underbrace{\psi_{h_1}^2}_{\in H^4(\overline{M}_{2,3})} \otimes \underbrace{\psi_{h_2} \cdot \kappa_3 \kappa_1^2}_{\in H^{12}(\overline{M}_{5,1})}}_{\in H^{16}(\overline{M}_\Gamma) = H^{16}(\overline{M}_{2,3} \times \overline{M}_{5,1})} \Big)$$

Thm The decorated strat. classes $[\Gamma, \alpha]$ form a finite generating set, as a $\mathbb{Q}$-vector space, of $RH^*(\overline{M}_{g,n})$.

Proof Let $S_{g,n} = \langle [\Gamma, \alpha] : \Gamma, \alpha \rangle_{\mathbb{Q}\text{-vs.}} \subseteq H^*(\overline{M}_{g,n})$.

Want to show: $S_{g,n} = RH^*(\overline{M}_{g,n})$. "$\subseteq$" have seen

"$\supseteq$": $(RH^*(\overline{M}_{g,n}))_{g,n}$ defined as smallest $\mathbb{Q}$-subalg of $H^*(-)$ closed under pushf. by ξ_Γ, π.

Suff. $S_{g,n}$ is syst. of $\mathbb{Q}$ - -

For this
- $S_{g,n}$ $\mathbb{Q}$-sub vector spaces ✓ $+$, $\lambda \cdot -$ $(\lambda \in \mathbb{Q})$
- contain $[\overline{M}_{g,n}] = [\text{triv. } \Gamma, 1]$
- closed under pushf. by ξ_{Γ_0}

$$[\Gamma_w, \alpha_w] \in RH^*(\overline{M}_{g(w),n(w)}) \quad , \quad w \in V(\Gamma_0)$$

$$\rightsquigarrow (\xi_{\Gamma_0})_* \Big(\prod \underline{[\Gamma_w, \alpha_w]} \Big) = [\Gamma', \alpha'] \in S_{g,n}$$

$$\rightsquigarrow (\xi_{\Gamma_0})_* \Big(\prod \underbrace{[\Gamma_w, \alpha_w]}_{\in S_{g(w),n(w)}} \Big) = [\Gamma', \alpha'] \in S_{g,n}$$

gluing Γ_w into vert. w of Γ_0

distrib. classes α_w to obtain decorat. on Γ'.

- Real work:
  - → closed under (cup) product. → $\boxed{\text{do below.}}$
  - → closed under forgetful pushforwards → (optional) Exercise.

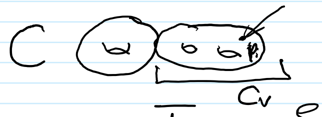
$\square$

$\rightsquigarrow$ How to write $[\Gamma_1, \alpha_1] \cdot [\Gamma_2, \alpha_2]$ as elem. of $S_{g,n}$, i.e. as linear comb. of $[\Gamma', \alpha']$?

Exercise Γ stable graph

→ $(\xi_\Gamma)^* \psi_i = \pi_v^* \psi_h \in H^*(\overline{\mathcal{M}}_\Gamma)$, $h \in H(\Gamma)$ corresp. to marking i, incident to $v \in V(\Gamma)$

$T^*_{p_i} C \cong T^*_{p_i} C_v$



→ $(\xi_\Gamma)^* \kappa_a = \sum_{v \in V(\Gamma)} \pi_v^* \kappa_a$

$$\overline{\mathcal{M}}_\Gamma \xrightarrow{\xi_\Gamma} \mathcal{M}_{g,n}$$

Fact (proved via deform. theory)

ξ_Γ is unramified, lci with

$$\mathcal{N}_{\xi_\Gamma} = \bigoplus_{\{h,h'\} \in E(\Gamma)} \mathbb{L}_h^\vee \otimes \mathbb{L}_{h'}^\vee .$$

rank $= \# E(\Gamma)$

, $\mathbb{L}_h$ cotan. line bdle on $\overline{\mathcal{M}}_{g(v),n(v)}$ ↺ half-edge h

line bdle on $\overline{\mathcal{M}}_\Gamma = \prod_v \overline{\mathcal{M}}_{g(v),n(v)}$

Prop Γ_1, Γ_2 stable gr. on $\overline{\mathcal{M}}_{g,n}$, $[\Gamma_2, \alpha]$ decorat. strat. class.
Then:

$$\underbrace{(\xi_{\Gamma_1})^* [\Gamma_2, \alpha]}_{\text{on } \overline{\mathcal{M}}_{g,n}} = \sum_{(\Gamma, \varphi_1, \varphi_2) \in \mathcal{G}_{\Gamma_1,\Gamma_2}} (\xi_{\varphi_1})_* \big((\xi_{\varphi_2}^* \alpha) \cdot \gamma_{ex} \big)$$

generic (Γ_1, Γ_2)-structures

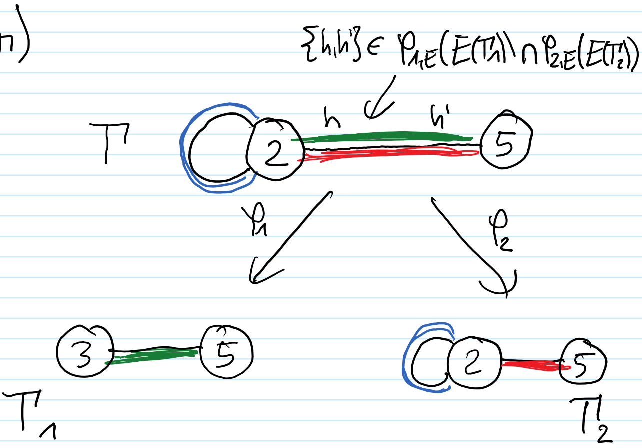
$$\begin{tikzcd} \displaystyle\coprod_{\substack{(\Gamma,\varphi_1,\varphi_2) \\ \in \mathcal{G}_{\Gamma_1,\Gamma_2}}} \overline{\mathcal{M}}_\Gamma \arrow[r, "\xi_{\varphi_2}"] \arrow[d, "\xi_{\varphi_1}"] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_{\Gamma_2} \ni \alpha \arrow[d, "\xi_{\Gamma_2}"] \\ \overline{\mathcal{M}}_{\Gamma_1} \arrow[r, "\xi_{\Gamma_1}"] & \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

where

$$\gamma_{ex} = \prod_{\substack{\{h,h'\} \in \\ \varphi_{1,E}(E(T_1')) \cap \varphi_{2,E}(E(T_2'))}} (-\psi_h - \psi_{h'}) \quad \in H^*(\overline{\mathcal{M}}_{T'})$$

and $(\xi_{T_1'})^*[T_2', \alpha]$ is in the taut. ring of $\overline{\mathcal{M}}_{T_1'}$.

$$\overline{\mathcal{M}}_{T_1'} \xrightarrow{\xi_{T_1'}} \overline{\mathcal{M}}_{g,n}$$



Proof Apply excess int. formula

⇝ Remains: $\gamma_{ex} = c_{top}(E)$, E excess bundle.

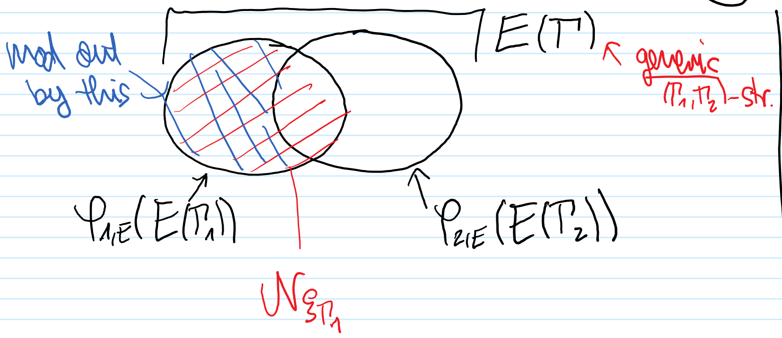
Using Fact above:

$$N_{\xi_{T_1'}} = \bigoplus_{\{h,h'\} \in E(T_1')} \mathbb{L}_h^\vee \otimes \mathbb{L}_{h'}^\vee$$

$$N_{\xi_{\varphi_2}} = \bigoplus_{\{h,h'\} \in E(T') \setminus \varphi_{2,E}(E(T_2'))} \mathbb{L}_h^\vee \otimes \mathbb{L}_{h'}^\vee$$

$$E \overset{\text{Def.}}{=} (\xi_{\varphi_1})^* N_{\xi_{T_1'}} / N_{\xi_{\varphi_2}}$$

$$= \bigoplus_{\substack{\{h,h'\} \in \\ \varphi_{1,E}(E(T_1')) \\ \cap\, \varphi_{2,E}(E(T_2'))}} \underbrace{\mathbb{L}_h^\vee \otimes \mathbb{L}_{h'}^\vee}_{\text{line bdls}}$$



$$c_{top}(E) = \prod_{\{h,h'\} \in \cdots \cap \cdots} c_1(\mathbb{L}_h^\vee \otimes \mathbb{L}_{h'}^\vee)$$

$$\overset{\text{Ex.}}{=} c_1(\mathbb{L}_h^\vee) + c_1(\mathbb{L}_{h'}^\vee) \overset{\text{Ex.}}{=} -c_1(\mathbb{L}_h) - c_1(\mathbb{L}_{h'}) \overset{\text{Def.}}{=} -\psi_h - \psi_{h'}$$

$$= \gamma_{ex}. \qquad \square$$

Cor The product of two dec. str. classes $[\Gamma_1,\alpha_1]\cdot[\Gamma_2,\alpha_2]$ on $\overline{\mathcal{M}}_{g,n}$ is given by

dec. strat. classes

$$[\Gamma_1,\alpha_1]\cdot[\Gamma_2,\alpha_2] = \sum_{(\Gamma,\varrho_1,\varrho_2)\in\mathcal{G}_{\Gamma_1,\Gamma_2}} \left[\Gamma, (\xi_{\varrho_1})^*\alpha_1\cdot(\xi_{\varrho_2})^*\alpha_2\cdot \gamma_{ex}\right]$$

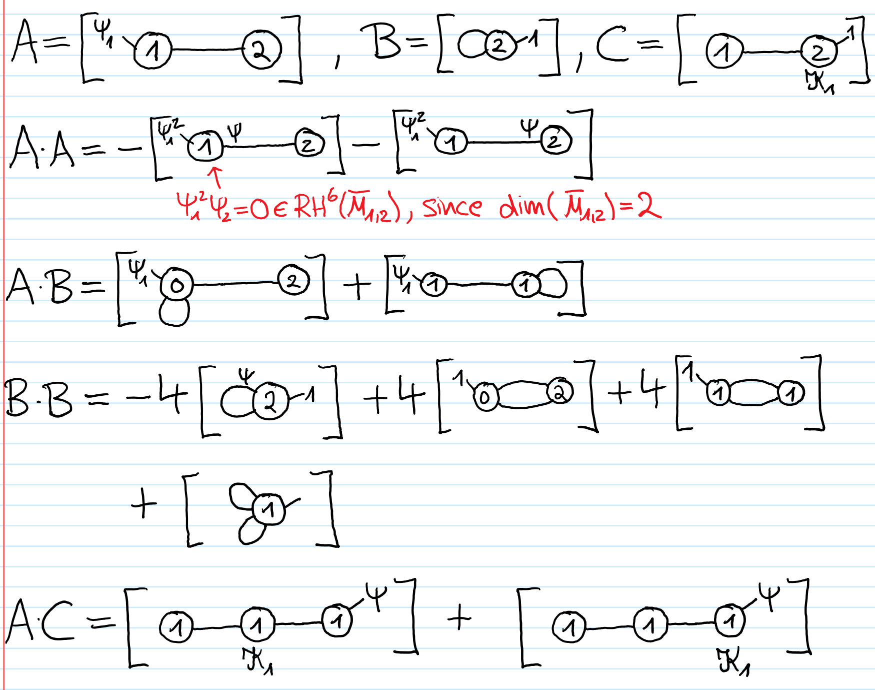
formula as before.

Proof

$$[\Gamma_1,\alpha_1]\cdot[\Gamma_2,\alpha_2] = \left((\xi_{\Gamma_1})_*\alpha_1\right)\cdot[\Gamma_2,\alpha_2]$$

proj. form → $= (\xi_{\Gamma_1})_*\Big(\underbrace{(\xi_{\Gamma_1})^*[\Gamma_2,\alpha_2]}_{\text{insert. formula from Prop. above.}}\cdot\alpha_1\Big) \overset{\text{proj. formulas.}}{=}$ formula above

$\square$



$\psi_1^2\psi_2 = 0 \in RH^6(\overline{\mathcal{M}}_{1,2})$, since $\dim(\overline{\mathcal{M}}_{1,2}) = 2$

Product in $RH^*(\overline{\mathcal{M}}_{g,n})$ described purely in terms of combinatorics of graphs.