Thm ([DM69, Kn83]) Let $2g-2+n>0$.

(a) There exist coarse moduli spaces $M_{g,n}$ and $\overline{M}_{g,n}$ for $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ / algebraic stacks $\mathcal{M}_{g,n}$, $\overline{\mathcal{M}}_{g,n}$

(b) They are normal alg. varieties (smooth stacks) of dimens. $3g-3+n$ and there exists inclus. $M_{g,n} \subset \overline{M}_{g,n}$ as nonempty open dense subvariety.

(c) $\overline{M}_{g,n}$ is irreducible, projective (proper) and has quotient singularities.

Zariski cover: V/G
V affine smooth var.
$G \curvearrowright V$ fin group.

(d) The boundary $\partial\overline{M}_{g,n} = \overline{M}_{g,n} \setminus M_{g,n}$ is the locus of singular curves, and a (Weil) divisor. / normal crossing Cartier divisor

(e) The locus $\overline{M}^{\circ}_{g,n} \overset{\text{open}}{\subseteq} \overline{M}_{g,n}$ of curves $(C, p_1, \dots, p_n)$ with trivial aut. group

$$\mathrm{Aut}(C, p_1, \dots, p_n) = \{\mathrm{id}_C\}$$

is a fine moduli space for the functor $\overline{\mathcal{M}}^{\circ}_{g,n}$ of st. curves without nontr. automorphisms. Thus $\exists$ universal family

$$\begin{array}{ccc} \overline{C}^{\circ}_{g,n} & & \\ \pi \downarrow \uparrow\uparrow p_1,\dots,p_n & & \\ \overline{M}^{\circ}_{g,n} & \subseteq & \overline{M}_{g,n} \end{array}$$

Construction of coarse moduli spaces

__Idea__ Moduli space of $\{C\}$ hard (don't have a lot of inform. to play with)

$\rightsquigarrow$ Start w/ space $\{(C, \text{extra data})\}$ $\rightsquigarrow$ quotient by choice of extra data.

__Auxiliary space for M_g__ Fix $K \geq 5$

$$K_g = \left\{ (C, \varphi: C \hookrightarrow \mathbb{P}^r) \;\middle|\; \begin{array}{l} C \text{ smooth, irred. proj. curve of genus } g, \\ \varphi \text{ non-degenerate embedding}, \\ \varphi^*\mathcal{O}(1) = \omega_C^{\otimes K} \end{array} \right\}$$

$\varphi(C)$ is not contained in any hyperplane in $\mathbb{P}^r$

where $\omega_C = \Omega^1_C$ is the cotangent bundle of C and

$$r = (2K-1)(g-1)-1.$$

__Why is K_g better than M_g?__

K_g parametrizes subschemes $C \subseteq \mathbb{P}^r$ in a fixed proj. space $\mathbb{P}^r$ (with some properties)

$\exists$ Grassmannian varieties $\hookrightarrow$ constructed by Grothendieck

$\rightsquigarrow$ K_g is locally closed subset of the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^r)$

smooth, irreducible open cond.

$\mathcal{O}(1)|_C \cong \omega_C^{\otimes k}$ closed condition

$\mathrm{Hilb}(\mathbb{P}^r) = \{(Z \subseteq \mathbb{P}^r) \mid Z \text{ subscheme}\}$

$\rightsquigarrow$ K_g gets a scheme structure

<u>How to recover M_g from K_g?</u> $(g \geq 2)^{k \geq 5}$

$$\deg(\omega_C^{\otimes k}) = k(2g-2) = 2k(g-1) > 2g+1$$

$\hookrightarrow$ $H^1(C, \omega_C^{\otimes k}) = 0$ (already seen)

$\hookrightarrow$ $\omega_C^{\otimes k}$ very ample $\rightsquigarrow$ complete lin. system defines an embedding into proj. space $\mathbb{P}^r$

$$h^0(C, \omega_C^{\otimes k}) \overset{\substack{\text{Riemann-Roch} \\ + h^1(\omega_C^{\otimes k})=0}}{=} 2k(g-1) + 1 - g = r+1$$

$\rightsquigarrow$ $\forall C$ smooth curve $\exists\, \varphi: C \hookrightarrow \mathbb{P}^r$ non-deg. embedd. w/ $\varphi^*(\mathcal{O}(1)) = \omega_C^{\otimes k}$

<u>Conversely</u> Such embeddings φ $\Longleftrightarrow$ $\underbrace{r+1 \text{ sections in } H^0(C, \omega_C^{\otimes k}) \leftarrow \dim r+1 \text{ non-degenerate} \Rightarrow \text{lin. indep.}}_{\text{specify a basis of } H^0(C, \omega_C^{\otimes k})}$

$$K_g = \left\{ (C, [s_0 : \dots : s_r]) \;\middle|\; \begin{array}{l} C \text{ smooth, gen. } g \\ s_0, \dots, s_r \in H^0(C, \omega_C^{\otimes k}) \text{ basis} \end{array} \right\}$$

The group PGL_{r+1} acts on K_g by fixing C, acting on the set of basis elements, action is simply transitive on the set of such choices

induced

$$PGL_{r+1} \curvearrowright \mathrm{Hilb}(\mathbb{P}^r) \longleftarrow PGL_{r+1} \curvearrowright \mathbb{P}^r$$

$\Rightarrow$ Expect: $\underset{\text{quotient (???)}}{K_g / PGL_{r+1}} \cong M_g = \{(C)\}$

<u>Problem</u> in general, it is not easy to define quotients in alg. geometry.

<u>Solution</u> Geometric invariant theory (GIT)

$\hookrightarrow$ invented by Mumford

$\hookrightarrow$ requires to check some conditions on K_g and action of PGL_{r+1} on it.

(points of K_g are semistable wrt the action of PGL_{r+1})

purely defined in terms of the group action.

What about stable curves?

C singular ⇝ Ω^1_C sheaf, not line bdle ⇝ no embedd. $C \to \mathbb{P}^r$ w/ $\mathcal{O}(1)|_C = \Omega^1_C$

⇝ replace Ω^1_C by dualizing sheaf ω_C

↳ exists for every proj. variety $X \rightsquigarrow \omega_X$

↳ Serre duality

$$H^i(X,\mathcal{F}) \cong H^{\dim X - i}(X, \mathcal{F}^\vee \otimes \omega_X)^\vee$$

↳ C nodal ⇝ ω_C line bundle

☆ C stable $\Leftrightarrow$ ω_C ample

↳ $\omega_C^{\otimes k}$ very ample for $k \geq 3$

Define

$$\overline{K}_g = \left\{ (C, \varphi : C \to \mathbb{P}^r) \;\middle|\; \begin{array}{l} C \text{ stable curve of genus } g \\ \varphi \text{ non-deg. embedding} \\ \varphi^*\mathcal{O}(1) = \omega_C^{\otimes k} \end{array} \right\}$$

fine moduli space

($C \subseteq \mathbb{P}^r$ has no automorphisms)

(☆) ⇝ every stable curve C appears in $\overline{K}_g$

⇝ can repeat argument from before, get $\overline{M}_g = \overline{K}_g / PGL_{r+1}$.

Get some more properties of $\overline{M}_g$:

semist. points wrt PGL_{r+1}-action

$$\begin{array}{ccc} \mathrm{Hilb}(\mathbb{P}^r)^{ss} & \supseteq & \overline{K}_g \\ \big\downarrow {\scriptstyle \exists\, \text{quot.}} & & \big\downarrow \\ \mathrm{Hilb}(\mathbb{P}^r)^{ss} /\!/ PGL_{r+1} & \supseteq & \overline{M}_g = \overline{K}_g / PGL_{r+1} \end{array}$$

closed subscheme

projective

also closed ⇒ projective.

Also get something for the stack $\overline{\mathcal{M}}_g$:

∃ family of curves over $\overline{K}_g$

$\Rightarrow$ $\overline{K}_g \longrightarrow \overline{\mathcal{M}}_g$ surjective, smooth, representable

smooth scheme

deformation theory

$\Downarrow$

$\overline{\mathcal{M}}_g$ is an algebraic stack

$\overline{\mathcal{M}}_g$ is smooth.

Dimension and smoothness

Goal Why $\dim \overline{\mathcal{M}}_{g,n} = 3g-3+n$?
(also: why $\overline{\mathcal{M}}_{g,n}$ smooth)

Use Deformation theory $/\mathbb{C}$

Given a scheme (or stack) $\mathcal{M}$, deformation theory studies morphisms $\mathrm{Spec}(A) \longrightarrow \mathcal{M}$, for A local Artinian $\mathbb{C}$-algebra.

A f.g. $\mathbb{C}$-algebra
A Artinian $\iff \dim_{\mathbb{C}} A < \infty$

$\mathcal{M}$ moduli space / stack
$\rightsquigarrow$ $\mathrm{Spec}(A) \longrightarrow \mathcal{M}$
$\updownarrow$
families over $\mathrm{Spec}(A)$

Exa $\mathbb{C}$ is trivial Artinian algebra.

$\mathbb{C}[X]/(f)$, $f \neq 0$ (f only one zero) otherw. not local

$A = \mathbb{C}[\varepsilon]/(\varepsilon^2)$ dual numbers

Exercise $\mathcal{M}$ complex scheme, $p : \mathrm{Spec}(\mathbb{C}) \to \mathcal{M}$ a $\mathbb{C}$-point.
Show that there exists an equivalence:

$$v \in T_p\mathcal{M} = \left(\mathfrak{m}_{\mathcal{M},p}/\mathfrak{m}_{\mathcal{M},p}^2\right)^\vee \longleftrightarrow \text{Morphisms} \quad \begin{tikzcd} \mathrm{Spec}(A) \arrow[r] & \mathcal{M} \\ \mathrm{Spec}(\mathbb{C}) \arrow[u, leftrightarrow, "\varepsilon \mapsto 0"] \arrow[ur, "p"'] & \end{tikzcd}$$

Zariski tangent space of $\mathcal{M}$ at p

$\mathcal{M} = \mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[x,y])$
$\mathrm{Spec}(\mathbb{C}) \longrightarrow \mathbb{A}^2$ w/ image $(0,0)$
$\mathbb{C} \longleftarrow \mathbb{C}[x,y]$
$0 \longmapsfrom x$
$0 \longmapsfrom y$

$V = (2,6) \in T_{(0,0)}\mathbb{A}^2$
$\mathbb{C} \xleftarrow{\varepsilon \mapsto 0} \mathbb{C}[\varepsilon]/(\varepsilon^2) \longleftarrow \mathbb{C}[x,y]$
$0 \longmapsfrom 2\varepsilon \longmapsfrom x$
$0 \longmapsfrom 6\varepsilon \longmapsfrom y$

$A = \mathbb{C}[\varepsilon]/(\varepsilon^2) \longrightarrow \mathbb{C}$
$\varepsilon \longmapsto 0$

So let's fix a smooth curve $C \in \mathcal{M}_g = \mathcal{M}$ and understand $T_C\mathcal{M}_g$ using the exercise.

$$\begin{tikzcd} & T_C\mathcal{M}_g \arrow[d, leftrightarrow] & \\ & \mathrm{Spec}(A) \arrow[r] \arrow[d, leftrightarrow] & \mathcal{M}_g \\ C \cong \mathcal{C}_0 \arrow[rr] \arrow[d] \arrow[drr, phantom, "\square"] & & \mathcal{C} \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[rr, hook] & & \mathrm{Spec}(A) \end{tikzcd}$$

family of smooth gen. g curves

Actually we can do story for more general varieties X:

Deformation of X over local Artinian ring A:
$$\begin{tikzcd} X \cong \mathcal{X}_0 \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d, "\text{flat, surjective}"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, hook] & \mathrm{Spec}(A) \end{tikzcd}$$

$\boxed{A = \mathbb{C}[\varepsilon]/(\varepsilon^2)}$: first order deformation.

For X smooth we make the following observations:

a) $(\varepsilon) \subset A$ unique prime ideal $\Rightarrow$ $\mathrm{Spec}(A) = \{pt\}$ as top. space

Similarly: $\mathcal{X}_0 \subseteq \mathcal{X}$ is cut out by square-zero ideal (ε)

$\underset{\text{Proposition}}{\rightsquigarrow}$ isomorphism of top. spaces

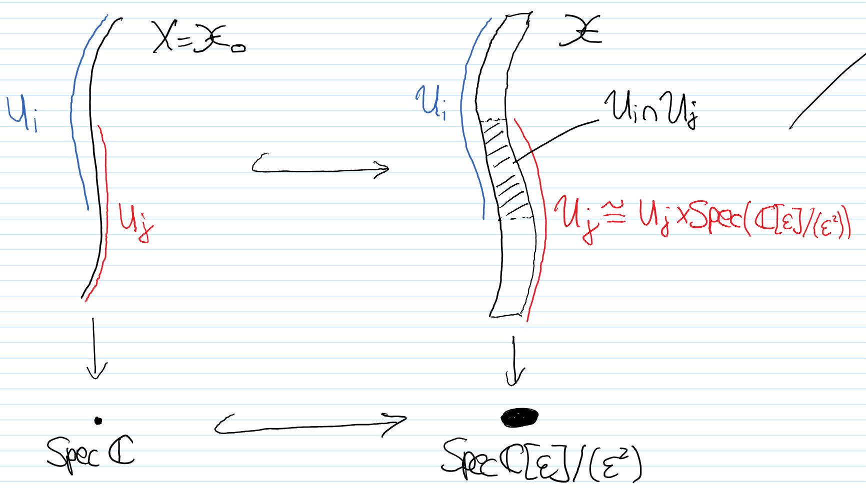
$U \subseteq \mathcal{X}_0 = X$ Zariski open $\xleftrightarrow{1:1}$ $\mathcal{U} \subseteq \mathcal{X}$ Zar. open.

Prop. U affine $\longleftrightarrow$ $\mathcal{U}$ affine

b) Try to go to affine open cover U_i of $X \leftrightarrow \mathcal{U}_i$ of $\mathcal{X}$.

Fact All first-order deformations of smooth affine schemes U are $\boxed{\text{trivial}}$:

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d] & \mathcal{U} \overset{*}{\cong} U \times \mathrm{Spec}(A) \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, hook] & \mathrm{Spec}(A) \end{tikzcd}$$



$0 \longleftarrow\!\shortmid\ \varepsilon$

deformation is trivial on each of pieces U_i of affine cover

$\Downarrow$

all information is in gluing data on overlaps $U_i \cap U_j$

$\varphi_{ij}: U_{ij} \times \mathrm{Spec}(A) \xrightarrow{\sim} U_{ij} \times \mathrm{Spec}(A)$

c) X separated $\rightsquigarrow$ U_{ij} also affine $\mathrm{Spec}(B)$

Prop

Isomorphisms φ_{ij}: $U_{ij} \times \mathrm{Spec}(A)$ ↺ $= \mathrm{Spec}(B[\varepsilon]/(\varepsilon^2))$

restrict. to $\mathrm{id}_{\mathrm{Spec}(B)}$ over $\mathrm{Spec}(\mathbb{C})$

$\updownarrow$

$B[\varepsilon]/(\varepsilon^2) \longrightarrow B[\varepsilon]/(\varepsilon^2)$

[illegible]

$$B[\varepsilon]/(\varepsilon) \longrightarrow B[\varepsilon]/(\varepsilon^2)$$

$$x + \varepsilon \cdot y \longmapsto x + \varepsilon(y + \eta_{ij}(x))$$

$$\eta_{ij} \in \mathrm{Der}_{\mathbb{C}}(B, B)$$ $\leftarrow$ $\mathbb{C}$-linear derivation.

$$\Updownarrow$$

$$V_{ij} \in H^0(U_{ij}, T_{U_{ij}})$$

<u>Next time</u>

$$(V_{ij})_{ij} \in \check{H}^1(X, T_X) \longleftrightarrow \text{data of deformation}$$

$$\Updownarrow \text{Exercise} \quad (C = X)$$

$$T_C \mathcal{M}_g$$

$$\Rightarrow T_C \mathcal{M}_g = H^1(C, T_C)$$ Exercise dim $H^1(C, T_C) = 3g - 3$