Lecture 15

Dienstag, 18. August 2020 09:40

$$T_C \mathcal{M}_g \cong H^1(C, T_C)$$

$$h^1(C, T_C) = h^1(C, \omega_C^\vee) \overset{S.D.}{=} h^0(C, (\omega_C^\vee)^\vee \otimes \omega_C)$$

$$= h^0(C, \omega_C^{\otimes 2}) = \deg(\omega_C^{\otimes 2}) + 1 - g$$

$$= 2\cdot(2g-2) + 1 - g = 3g-3 \qquad \square$$

Some variants & extensions

Pointed curves

$$T_{(C,p_1,\dots,p_n)} \mathcal{M}_{g,n} \cong H^1(C, T_C(-p_1 - \dots - p_n))$$

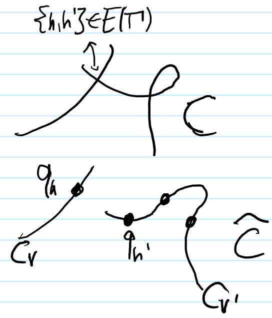
$\hookrightarrow \dim = 3g-3+n$

Stable curves

First order deform. of $C \cong \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)$

Γ stable graph of C and

$(C_v, (q_h)_{h\in H(v)})_{v\in V(\Gamma)}$ preim. of C under ξ_Γ



Then we have exact sequence:

$$0 \to \underbrace{\bigoplus_{v\in V(\Gamma)} H^1(C_v, T_{C_v}(-\textstyle\sum_{h\in H(v)} q_h))}_{\text{locally triv. deform.}} \to \underbrace{\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)}_{\text{all deform. of } C} \to \underbrace{\bigoplus_{\{h,h'\}\in E(\Gamma)} T_{q_h}(C_v) \otimes T_{q_{h'}}(C_{v'})}_{\text{detects smoothing of nodes}} \to 0$$

$$0 \to T_{(C_v,(q_h))_v} \overline{\mathcal{M}}_\Gamma \longrightarrow T_C(\overline{\mathcal{M}}_{g,n}) \longrightarrow \bigoplus_{\{h,h'\}\in E(\Gamma)} \mathbb{L}_h^\vee \otimes \mathbb{L}_{h'}^\vee \Big|_{(C_v,(q_h))}$$

$\mathcal{N}_{\xi_\Gamma}\Big|_{(C_v,(q_h))_v}$

line bdles on $\overline{\mathcal{M}}_\Gamma$

Higher order deformations and obstructions

<u>Claim</u> deform. over more general Artin. $\mathbb{C}$-alg. A allow us to detect smoothness

<u>Def</u> (Formal smoothness)

$\mathcal{M}$ scheme (stack) over $\mathbb{C}$ is <u>formally smooth</u> if for every surjective map $A' \to A$ of Artinian local $\mathbb{C}$-algebras with kernel $I \subseteq A'$ (s.t. $A = A'/I$) of dimension 1 and every map $\mathrm{Spec}(A) \to \mathcal{M}$ $\exists$ map $\mathrm{Spec}(A') \to \mathcal{M}$ s.t.

$$\begin{tikzcd} \mathrm{Spec}(A) \arrow[r, hook, "\text{closed embedd.}"] \arrow[d] & \mathrm{Spec}(A') \arrow[dl, dashed, "\exists"] \\ \mathcal{M} & \end{tikzcd}$$

commutes.

$\mathcal{M} = \mathcal{M}_{g,n}$

$$\begin{tikzcd} \mathcal{C}_A \arrow[r, dashed, "\exists?"] \arrow[d] & \mathcal{C}_{A'} \arrow[d, dashed] \\ \mathrm{Spec}(A) \arrow[r, hook] & \mathrm{Spec}(A') \end{tikzcd}$$

↳ Yes! this works more generally

For X smooth variety, such that $H^2(X, T_X) = 0$, any deformation $\mathcal{X}_A \to \mathrm{Spec}(A)$ can be extended to ⇝ X is unobstructed.

$$\begin{array}{ccc} \mathcal{X}_A & \to & \mathrm{Spec}(A) \\ \cap & & \cap \\ \mathcal{X}_{A'} & \to & \mathrm{Spec}(A'). \end{array}$$

$X = C$ smooth curve $\Rightarrow H^2(C, T_C) = 0$ for dim. reasons
$\Rightarrow \mathcal{M}_g$ is formally smooth at C.

Infinitesimal automorphisms

$H^0(X, T_X) \longleftrightarrow$ infinit. automorph. of X
$H^1(X, T_X) \longleftrightarrow$ deformations of X
$H^2(X, T_X) \longleftrightarrow$ obstructions to deform. of X

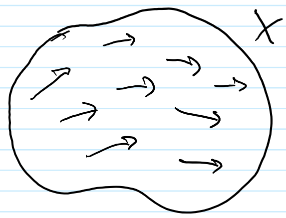
$$\mathrm{Aut}(X) \hookrightarrow \mathrm{Hilb}(X \times X), \quad \varphi \longmapsto (\Gamma_\varphi \subseteq X \times X)$$

$\mathrm{Aut}(X) = \{\varphi : X \to X \text{ automorph.}\}$

← embeds $\mathrm{Aut}(X)$ as closed pts. of subscheme of $\mathrm{Hilb}(X \times X)$
$\Downarrow$
$\mathrm{Aut}(X)$ scheme structure.

Then show (using def. th. of $\mathrm{Hilb}(X \times X)$):

$$T_{\mathrm{id}_X} \mathrm{Aut}(X) = H^0(X, T_X)$$ ← infin. autom.



Connects to autom. gps of smooth curves:
C smooth genus g

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$$\dim H^0(C, T_C) = \begin{cases} 3 & , g=0 \\ 1 & , g=1 \\ 0 & , g\geq 2 \end{cases} \quad \begin{array}{l} \longleftrightarrow \mathrm{Aut}(C) \cong PGL_2 \quad \text{3-dim.} \\ \longleftrightarrow \mathrm{Aut}(C)^0 \cong C \ (\text{translat.}) \quad \text{1-dim.} \\ \longleftrightarrow \mathrm{Aut}(C) \text{ discrete} \quad \text{0-dim.} \end{array}$$

Exerc. Use def. thy to compute $\dim \mathbb{P}^n = n$.

Exerc. - - - - - - - - - $\dim^{\mathrm{To}} \mathrm{Jac}(C) = g$.

C smooth curve, genus g $\qquad \mathrm{Jac}(C) = \mathrm{Pic}^0_{C/\mathbb{C}}$

$\mathrm{Jac}(C)$ algebraic group & char. 0

$\Rightarrow \mathrm{Jac}(C)$ smooth.

$\Rightarrow \dim \mathrm{Jac}(C) = g$

Density of locus M_g of smooth curves & local stud. of boundary

Start with proving: $M_{g,n} \subseteq \overline{M}_{g,n}$ is open

Prop. Let $\pi: \mathcal{C} \to B$ be a family of stable curves, then locus $B_0 \subseteq B$ of $b \in B$ w/ $\mathcal{C}_b$ smooth is open in B!

Proof Locus $\mathcal{C}^{sm} \subseteq \mathcal{C}$ of pts where π is smooth is open

$\rightsquigarrow \mathrm{Sing}(\pi) = \mathcal{C} \setminus \mathcal{C}^{sm}$ is closed in $\mathcal{C}$

$\xrightarrow{\pi \text{ proper}} \pi(\mathrm{Sing}(\pi)) \subseteq B$ closed

$\Rightarrow B_0 = B \setminus \pi(\mathrm{Sing}(\pi))$ open. $\square$

Cor $i: M_{g,n} \longrightarrow \overline{M}_{g,n}$ representable, open embedd.

Pf

$$\begin{array}{ccc} M_{g,n} & \xrightarrow{i} & \overline{M}_{g,n} \\ \uparrow & & \uparrow \\ B_0 & \xrightarrow{i'} & B \text{ scheme} \end{array}$$

B_0 scheme $\rightsquigarrow i$ represent.

i' open embedding.

$\Rightarrow i$ open embedd. $\square$

$M_{g,n} \neq \emptyset$

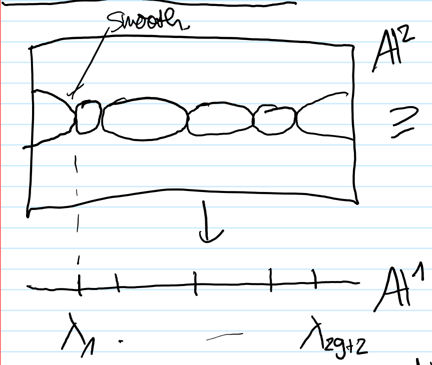
If we show that $\forall g \geq 0\ \exists C$ smooth curve of genus g

$\rightsquigarrow (C, p_1, \dots, p_n) \in M_{g,n}$ $\quad 2g-2+n>0$ (Fact about automorph.)

To construct C

To construct C

$(y^2=x)$ smooth at $(0,0)$



$\supseteq (y^2=(x-\lambda_1)\cdots(x-\lambda_{2g+2}))$

smooth affine curve $C_0 \overset{\exists!}{\subseteq} C$ → genus g hyperell. curve

smooth, projective curve

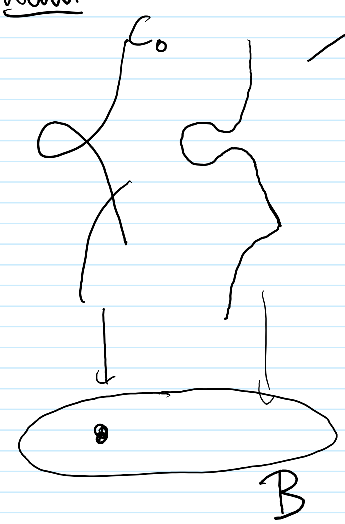
← distinct pts.

$\rightsquigarrow \mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$ nonempty open subset

If we knew that $\overline{\mathcal{M}}_{g,n}$ is irreducible $\Rightarrow \mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$ is dense.

try to show directly!

Want



want to approximate C_0 by smooth curves.

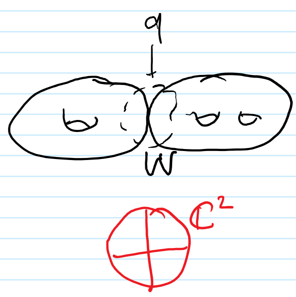
1 variant via ideas from deformation theory

$\rightsquigarrow B = \operatorname{Spec} \mathbb{C}[[t_1,\dots,t_{3g-3}]]$

[Deligne-Mumford]

Explain story in complex-analytic language

Assume C_0 has single pole q



$\rightsquigarrow \exists$ open neighbourhood W of q of the form

def. of node

$$W \cong \{(x,y) \in B_\varepsilon(\mathbb{C}^2) \mid x \cdot y = 0\}$$

There exists a family

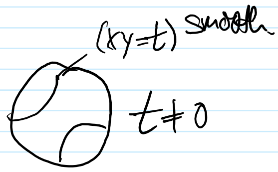
$(xy=t)$ smooth

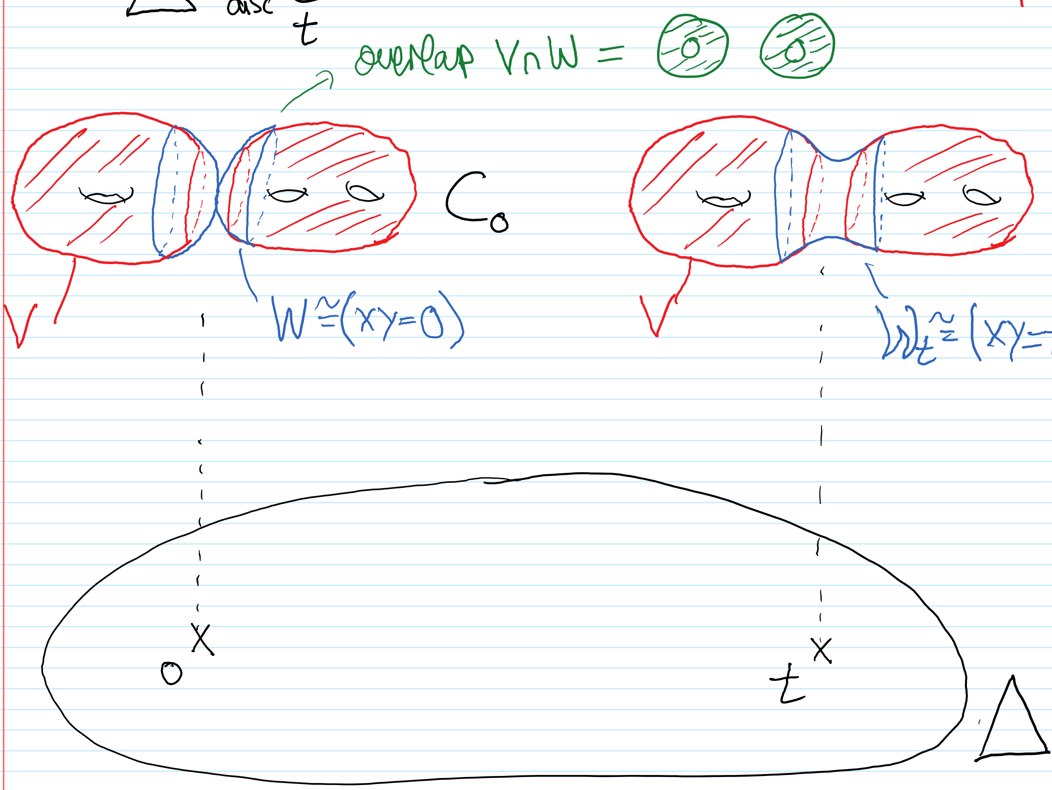
There exists a family

$$\mathcal{W} = \{ (x,y,t) \in B_\varepsilon(\mathbb{C}^2)\times\Delta \mid x\cdot y = t \}$$

$$\mathcal{W} \to \Delta \underset{\text{disc}}{\subseteq} \mathbb{C}_t$$

family smoothing node q over Δ

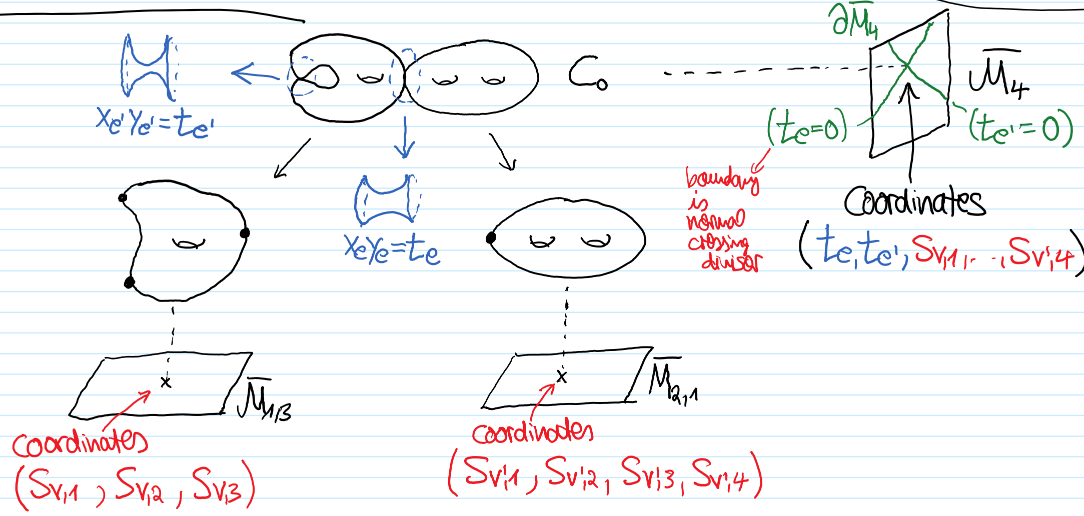




get global def. of C_0 over Δ

can upgrade this picture to get local (holom.) coordinates around each point $C_0 \in \overline{M}_{g,n}$

For local coordinates:



Def The completed local ring $\widehat{\mathcal{O}}_{C_0, \overline{\mathcal{M}}_{g,n}}$ is the compl. loc. ring $\widehat{\mathcal{O}}_{C', U}$ for

$$\begin{tikzcd} U \arrow[r, "\text{étale}"] & \overline{\mathcal{M}}_{g,n} \\ C' \arrow[u, "\text{scheme}"] \arrow[r] & C_0 \arrow[u] \end{tikzcd}$$

Theorem ([DM])

$C_0 \in \overline{\mathcal{M}}_{g,n}$ closed pt, $\Gamma' = \Gamma'(C_0)$, $(C_v, (q_h))_v$ preim. of C_0 under $\xi_{\Gamma'}$

Then we have an isomorphism

$$\widehat{\mathcal{O}}_{C_0, \overline{\mathcal{M}}_{g,n}} \cong \mathbb{C}[[t_e, s_{v,j} : e \in E(\Gamma'), v \in V(\Gamma'), j = 1, .., 3g(v)-3+n(v)]]$$

Under natural map $\operatorname{Spec} \widehat{\mathcal{O}}_{C_0, \overline{\mathcal{M}}_{g,n}} \longrightarrow \overline{\mathcal{M}}_{g,n}$

$$\bigcup_{e \in E(\Gamma')} V(t_e) \quad \cdots\cdots \quad \partial \overline{\mathcal{M}}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$$

union of coordinate hyperplanes.

preim. of $\mathcal{M}_{g,n}$ = complem. of $\bigcup_e V(t_e)$

dense in $\operatorname{Spec} \widehat{\mathcal{O}}_{C_0, \overline{\mathcal{M}}_{g,n}}$

$\Rightarrow \mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$ is dense.

$\rightsquigarrow$ $\partial \overline{\mathcal{M}}_{g,n}$ is normal crossings divisor.

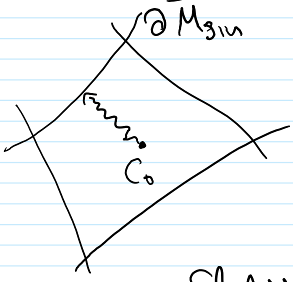
On the other hand, for $\xi_{\Gamma'} : \overline{\mathcal{M}}_{\Gamma'} \longrightarrow \overline{\mathcal{M}}_{g,n}$

$$\begin{tikzcd} & \overline{\mathcal{M}}_{g,n} \\ \operatorname{Spec} \widehat{\mathcal{O}}_{(C_v, (q_h)), \overline{\mathcal{M}}_{\Gamma'}} \arrow[r] & \operatorname{Spec} \widehat{\mathcal{O}}_{C_0, \overline{\mathcal{M}}_{g,n}} \arrow[u] \end{tikzcd}$$

(generators of $\widehat{\mathcal{O}}_{(C_v, (q_h)), \overline{\mathcal{M}}_{g(v), n(v)}}$) $\longleftarrow s_{v,j}$

Missing: $\overline{M}_{g,n}$ is irreducible & proper

$M_{g,n}$ smooth: connected.



Show $C_0 \overset{\text{deg.}}{\rightsquigarrow} C_0' \in \partial\overline{M}_{g,n}$

$\uparrow$

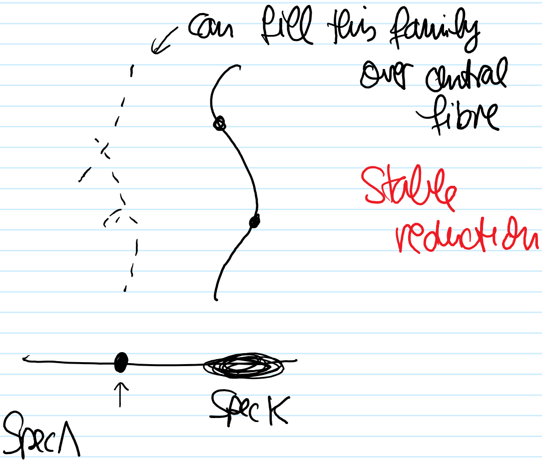
$\underbrace{\coprod \overline{M}_{g(v),n(v)}}_{\text{connected}}$

indust.

$\rightsquigarrow \partial\overline{M}_{g,n}$ connected

properness:

Valuative criterion of properness



Λ valuation ring