2) Fine and coarse moduli spaces

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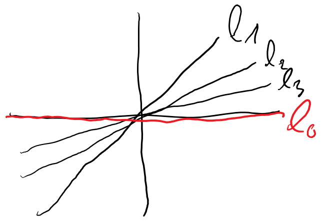
2.1) Motivation

$\rightsquigarrow$ What does it mean ~~that~~ a variety has "natural structure" for being a moduli space of some geom. objects.

Exa $\mathbb{P}^n$ projective space
$\rightsquigarrow$ "parametrizing lines through origin in $\mathbb{C}^{n+1}$"

Topology



Sequence

$$l_n = \left\langle \begin{pmatrix} 1 \\ 1/n \end{pmatrix} \right\rangle \subseteq \mathbb{C}^2$$

$$\Big\downarrow n \to \infty$$

$$l_0 = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \subset \mathbb{C}^2$$

$\rightsquigarrow$ True that $\underset{\mathbb{P}^1}{\underset{\cap}{(}}[l_n]) \xrightarrow{n\to\infty} [l_0]$ in $\mathbb{P}^n(\mathbb{C})$

Algebraic geometry

$$L_t = \left\langle \begin{pmatrix} 1 \\ t \end{pmatrix} \right\rangle \subseteq \mathbb{C}^2$$

"depend algebraically on $t \in \mathbb{C}$"

$$\rightsquigarrow \quad \mathbb{A}^1 \longrightarrow \mathbb{P}^1 \qquad \text{algebraic morphism } (\checkmark)$$
$$t \longmapsto [L_t]$$

More generally: For X any scheme over $\mathbb{C}$ there should be a correspondence

$$\text{Families of lines param. by } X \longleftrightarrow \text{Morphisms } X \to \mathbb{P}^n \qquad (*)$$

$$(L_t)_{t\in X} \longleftrightarrow (t \mapsto [L_t])$$

Crucial insight: Knowing the RHS of $(*)$ uniquely determines the scheme $\mathbb{P}^n$.

2.2) Moduli functors and fine moduli spaces.

$\rightsquigarrow$ Right language: Category theory.

$Sch_{\mathbb{C}}$: category of schemes / $\mathbb{C}$

Obj: ______

Morph. morph. of schemes over $\mathbb{C}$

Given $M \in Sch_{\mathbb{C}}$ there is a functor

$$h^M : Sch_{\mathbb{C}}^{op} \longrightarrow Sets$$
$$X \longmapsto Mor_{\mathbb{C}}(X, M)$$

$\rightsquigarrow$ RHS of (*)

$\rightsquigarrow$ contravariant functor (without op)

$$X' \xrightarrow{g} X \rightsquigarrow h^M(X) \longrightarrow h^M(X')$$
$$(X \xrightarrow{f} M) \longmapsto (X' \xrightarrow{g} X \xrightarrow{f} M)$$

Lemma (Yoneda's lemma)

The functor

$$h^{-} : Sch_{\mathbb{C}} \longrightarrow Functors(Sch_{\mathbb{C}}^{op} \to Sets)$$
$$M \longmapsto h^M$$

Category
Obj: $h: Sch_{\mathbb{C}}^{op} \to Sets$
Mor: natural transform.

is a fully faithful embedding. In other words:

→ given schemes $M, N \in Sch_{\mathbb{C}}$, the morphisms $M \to N$ are in bijection w/ natural transform. $h^M \to h^N$

→ in particular: $M \cong N$ if and only if $h^M \cong h^N$.

Def A moduli functor is a functor $h: Sch_{\mathbb{C}}^{op} \to Sets$.

Consists of:

→ for every scheme X, a set $h(X)$ "families param." by X

→ for every morph. $f: X' \to X$, a map $h(X) \xrightarrow{h(f)} h(X')$ "pullback of families" under f

→ satisfying → $h(id_X) = id_{h(X)}$
→ $h(f \circ g) = h(f) \circ h(g)$

"compatibility of pullback w/ id, compos."

Def A moduli functor h is called <u>representable</u> if $h \cong h^M$ for some $M \in Sch_{\mathbb{C}}$, M is called a <u>fine moduli space</u> for h.

Ex Show " $\mathbb{P}^n$ moduli space of lines in $\mathbb{C}^{n+1}$ thr. 0 ".

Def Families of lines

line bdle over X — trivial vect. bdle / X

$$h(X) = \left\{ \begin{array}{c} L \xrightarrow{i} X \times \mathbb{C}^{n+1} \\ \downarrow \quad \swarrow \\ X \end{array} \;\middle|\; \begin{array}{l} i \text{ is injective} \\ \text{map of vector} \\ \text{bundles} \end{array} \right\} / \text{isom}$$

$(L \xrightarrow{i} X \times \mathbb{C}^{n+1}) \overset{\text{isom}}{\sim} (L' \xrightarrow{i'} X \times \mathbb{C}^{n+1})$ iff. $\exists$

$$\begin{tikzcd} & X \times \mathbb{C}^{n+1} & \\ L \arrow[ur] \arrow[rr, dashed, "\exists", "\sim"'] \arrow[dr] & & L' \arrow[ul] \arrow[dl] \\ & X & \end{tikzcd}$$

commutes

Pullback $f : X' \longrightarrow X$

$$h(f) : h(X) \longrightarrow h(X')$$

$$(L \xrightarrow{i} X \times \mathbb{C}^{n+1}) \longmapsto (f^* L \xrightarrow{f^*(i)} X' \times \mathbb{C}^{n+1})$$

<u>Check</u> satisfies the compatibility

<u>Now</u> Construct natural equivalence $h \cong h^{\mathbb{P}^n}$

$\rightsquigarrow$ for every scheme X a bijection

$$h(X) \xrightarrow{\sim} h^{\mathbb{P}^n}(X) = Mor(X, \mathbb{P}^n)$$

$$\begin{array}{c} L \xrightarrow{i} X \times \mathbb{C}^{n+1} \\ \downarrow \quad \swarrow \\ X \end{array} \longmapsto \left(\begin{array}{c} X \longrightarrow \mathbb{P}^n \\ x \longmapsto [i(L_x)] \\ \quad\quad \cap \\ \quad\quad x \times \mathbb{C}^{n+1} \end{array} \right)$$

$\rightsquigarrow$ such that for $f : X' \longrightarrow X$

$$\begin{tikzcd} h(X) \arrow[r, "\sim"] \arrow[d, "h(f)"'] & h^{\mathbb{P}^n}(X) \arrow[d, "h^{\mathbb{P}^n}(f)"] \\ h(X') \arrow[r, "\sim"] & h^{\mathbb{P}^n}(X') \end{tikzcd}$$

<u>To construct this map</u>:

total spaces

$$\left(\begin{array}{c} L \xrightarrow{i} X \times \mathbb{C}^{n+1} \\ \downarrow \quad \swarrow \\ X \end{array} \right) \in h(X)$$

natural equival. $\rightsquigarrow$ $\Updownarrow$ $\rightsquigarrow$ going from tot. space to loc. free sheaves

inier.

.

$$\left(\begin{tikzcd} \mathcal{L} \arrow[r, hook, "i", "\text{injec.}"'] \arrow[d] & \mathcal{O}_X^{\oplus n+1} \arrow[dl] \\ X & \end{tikzcd}\right)$$

$\Updownarrow$ $\rightarrow$ dualize

$$\left(\begin{tikzcd} \mathcal{L}^\vee \cong \mathcal{M} \arrow[d] & \mathcal{O}_X^{\oplus n+1} \arrow[l, two heads] \arrow[dl] \\ X & \end{tikzcd}\right)$$

$\Updownarrow$

$$\left(\begin{array}{c} \mathcal{M} \\ \downarrow \\ X \end{array}, \quad \underbrace{s_0, \dots, s_n}_{\text{sections}} \in H^0(X, \mathcal{M}) \;:\; \begin{array}{l} s_0, \dots, s_n \\ \text{don't vanish} \\ \text{simultaneously} \end{array}\right)$$

$\Updownarrow$

$$\left(X \xrightarrow{g} \mathbb{P}^n\right) \in h^{\mathbb{P}^n}(X)$$

"$\Uparrow$" $\mathcal{M} = g^* \mathcal{O}_{\mathbb{P}^n}(1)$

$s_i = g^* x_i$

$H^0(\mathbb{P}^n, \mathcal{O}(1)) = \langle x_0, \dots, x_n \rangle$

"$\Downarrow$" Choose trivializing open cover of X for $\mathcal{M}$

define $X \longrightarrow \mathbb{P}^n$

$x \longmapsto [s_0(x) : \dots : s_n(x)]$

s_i are functions

$\rightsquigarrow$ check: independ. of trivializ.

agree on overlaps.

$\Rightarrow$ $\mathbb{P}^n$ is fine moduli space for h.

<u>Def</u> Given a functor h and natural isomorphism $h \cong h^M$, $M \in Sch_{\mathbb{C}}$ we define the <u>universal family</u> $U \in h(M)$ to be the unique elem. in $h(M)$ corresp. to $id_M \in h^M(M) \cong h(M)$

Exerc. W/ notat. as Def

(a) For X scheme, $F \in h(X)$ family$/X$, $\exists!$ morph. $f: X \to M$ s.t. $f^* U = F$.

(b) For h from Example above

$$\begin{tikzcd} L = \{(l,v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid v \in l\} \arrow[r, "i", hook] \arrow[d] & \mathbb{P}^n \times \mathbb{C}^{n+1} \arrow[dl] \\ \mathbb{P}^n & \end{tikzcd}$$

is universal family.

$L \cong \mathcal{O}_{\mathbb{P}^n}(-1)$.

(c) For interm. functor:

$(\mathcal{O}_{\mathbb{P}^n}(1), x_0, \ldots, x_n)$ is universal family.

Exerc Fibre product $X \times_Z Y \in Sch_{\mathbb{C}}$ is fine moduli space for functor:

$$h^X \times_{h^Z} h^Y (S) = \left\{ \begin{tikzcd} S \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd} \right\}$$

$\rightsquigarrow$ univ. family?

2.3 Application: The Picard scheme

Picard group of Y:

$\text{Pic}(Y) = \{ \mathcal{L} : \mathcal{L} \text{ line bdle on } Y \} / \text{isom.}$ $\rightsquigarrow$ Set $\rightsquigarrow$ group wrt. $\otimes$

$\rightsquigarrow$ associated moduli fct, moduli space of line bdles?

Def (absolute Picard functor)

Given $X \in Sch_{\mathbb{C}}$, a family of line bdl. on Y param. by X:

$$\text{Pic}_Y(X) = \text{Pic}(X \times Y).$$

Given $f: X' \to X$, define morph. $\text{Pic}_Y(X) \longrightarrow \text{Pic}_Y(X')$

$\mathcal{L} \longmapsto (f \times id_Y)^* \mathcal{L}$

$\rightsquigarrow$ defines a moduli functor Pic_Y

Prop Pic_Y never representable for $Y \neq \emptyset$.

Proof X scheme w/ nontriv. line bdle $\mathcal{M}$ (e.g. $X = \mathbb{P}^1$, $\mathcal{M} = \mathcal{O}(1)$)

$$\rightsquigarrow \exists s \begin{tikzcd} X \times Y \arrow[d, "\pi_X"] \\ X \arrow[u, bend left, "s"] \end{tikzcd} \qquad \begin{matrix} \mathcal{M}_X = \pi_X^*(\mathcal{M}) \in \mathrm{Pic}_Y(X) \\ \vdots \\ \mathcal{M} \end{matrix}$$

$Y \neq \emptyset$

not trivial elem. $\mathcal{O}_{X \times Y} \in \mathrm{Pic}_Y(X)$

because $s^* \mathcal{M}_X = \mathcal{M} \neq s^* \mathcal{O}_{X \times Y} = \mathcal{O}_X$.

Given open cover $\bigcup_i U_i = X$ trivializing $\mathcal{M}$: $\mathcal{M}|_{U_i} = \mathcal{O}_{U_i}$

$$\Rightarrow \mathcal{M}_X|_{U_i \times Y} \cong \mathcal{O}_{U_i \times Y}$$

$$(U_i \hookrightarrow X)^* \mathcal{M}_X$$

map given by $\mathcal{M}_X$

$$\begin{tikzcd} X \arrow[r] \arrow[drr, dashed, no head] & \mathrm{Pic}_Y \\ \coprod_i U_i \arrow[u] \arrow[r] & \mathrm{pt} \arrow[u] \end{tikzcd}$$

glue maps along open cover

map given by $\mathcal{O}_{\mathrm{pt} \times Y} \in \mathrm{Pic}_Y(\mathrm{pt})$

If $\exists$ moduli space Pic_Y

$$\Rightarrow \begin{tikzcd} X \arrow[r] \arrow[rr, bend left] & \mathrm{pt} \arrow[r] & \mathrm{Pic}_Y \end{tikzcd}$$

$$\mathcal{O}_{X \times Y} \text{ - - - } \mathcal{O}_{\mathrm{pt} \times Y} \text{ - - - } \mathcal{U} \in \mathrm{Pic}(\mathrm{Pic}_Y \times Y)$$

univ. line bdle

$$\neq$$

by def. $\mathcal{M}_X$ - - - - - - $\mathcal{U}$

$\rightsquigarrow$ Contradiction. $\square$