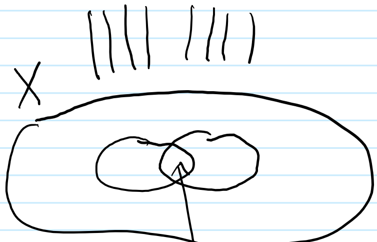
Lecture 3

Montag, 4. Mai 2020 16:33

<u>Rmk</u> Slogan:

"Existence of automorph. prevents exist. of" fine moduli spaces



glue in nontriv. way on overlaps

<u>Def</u> (relative Picard functor) Y scheme

$$\mathrm{Pic}_{Y/\mathbb{C}} : \mathrm{Sch}^{op} \longrightarrow \mathrm{Sets}$$

$$X \longmapsto \mathrm{Pic}(X \times Y) / \pi_X^* \mathrm{Pic}(X)$$

$\pi_X : X \times Y \to X.$

$\longrightarrow$ pullback via pullb. of line bdles

<u>Thm</u> Y integral, proj. variety $/\mathbb{C}$, then the functor $\mathrm{Pic}_{Y/\mathbb{C}}$ is representable by a separated, loc. finite type scheme

$$\mathbb{P}\mathrm{ic}_{Y/\mathbb{C}}$$

<u>Rmks</u>

$\rightarrow$ $\mathrm{Pic}_{Y/\mathbb{C}}(\mathrm{Spec}\,\mathbb{C}) = \mathrm{Pic}(Y) / \pi_{pt}^* (\underbrace{\mathrm{Pic}(pt)}_{\text{triv.}}) = \mathrm{Pic}(Y)$

$\Rightarrow$ $\mathbb{P}\mathrm{ic}_{Y/\mathbb{C}}(\mathbb{C}) = \mathrm{Pic}(Y)$

$\rightarrow$ A line bundle $\mathcal{L}$ on $\mathbb{P}\mathrm{ic}_{Y/\mathbb{C}} \times Y$ represent. univ. family is called a <u>Poincaré line bunde</u>.

$$\begin{tikzcd}
\mathcal{L} & & \mathcal{L}_0 \\
\mathbb{P}\mathrm{ic}_{Y/\mathbb{C}} \times Y \arrow[d] & & Y \arrow[ll, hook'] \arrow[d] \\
\mathbb{P}\mathrm{ic}_{Y/\mathbb{C}} & & {[\mathcal{L}_0] = pt} \arrow[ll, hook']
\end{tikzcd}$$

$\to$ $\mathrm{Pic}(Y)$ has structure of abelian group ($\otimes$)

$\Rightarrow$ $\mathbb{P}\mathrm{ic}_{Y/\mathbb{C}}$ has str. of abelian <u>group scheme</u>

$$\mathbb{P}\mathrm{ic}_{Y/\mathbb{C}} \times \mathbb{P}\mathrm{ic}_{Y/\mathbb{C}} \longrightarrow \mathbb{P}\mathrm{ic}_{Y/\mathbb{C}} \qquad \mathbb{P}\mathrm{ic}_{Y/\mathbb{C}} \longrightarrow \mathbb{P}\mathrm{ic}_{Y/\mathbb{C}}$$

$$([\mathcal{L}], [\mathcal{M}]) \longmapsto [\mathcal{L} \otimes \mathcal{M}] \qquad [\mathcal{L}] \mapsto [\mathcal{L}^\vee]$$

are alg. morphism.

$\to$ $\mathbb{P}\mathrm{ic}^0_{Y/\mathbb{C}} \subseteq \mathbb{P}\mathrm{ic}_{Y/\mathbb{C}}$ conn. comp. of $[\mathcal{O}_Y]$

$\rightsquigarrow$ subgroup of $\mathbb{P}\mathrm{ic}_{Y/\mathbb{C}}$

<u>Exa</u> $Y = C$ smooth, proj. irred. curve

$\Rightarrow$ $\mathbb{P}\mathrm{ic}^0_{C/\mathbb{C}} = \mathrm{Jac}(C)$ <u>Jacobian of C</u>

<u>Exa</u>

(a) $Y = pt \rightsquigarrow \mathrm{Pic}(pt) = \{\mathcal{O}\}$, indeed $\mathbb{P}\mathrm{ic}_{pt/\mathbb{C}} = pt. = \mathrm{Spec}\,\mathbb{C}$

X scheme: $\mathbb{P}\mathrm{ic}_{pt/\mathbb{C}}(X) = \mathrm{Pic}(X)/\pi_X^*\mathrm{Pic}(X) = \{[\mathcal{O}_X]\}$ $\updownarrow$ nat. isom.

$h^{pt}(X) = \{X \to pt\}$

(b) $Y = \mathbb{P}^n \rightsquigarrow \mathrm{Pic}(\mathbb{P}^n) = \{\mathcal{O}(n) : n \in \mathbb{Z}\}$

$$\mathbb{P}\mathrm{ic}_{\mathbb{P}^n/\mathbb{C}} = \coprod_{n \in \mathbb{Z}} \mathrm{Spec}(\mathbb{C}) \quad - - - - - \quad . \; . \; .$$

(c) $Y = \mathbb{A}^n \rightsquigarrow \mathrm{Pic}(\mathbb{A}^n) = \{\mathcal{O}_{\mathbb{A}^n}\}$

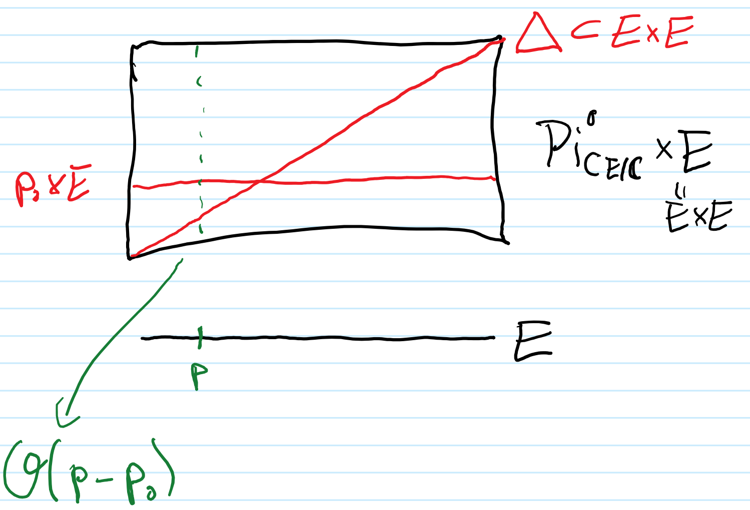
actually: $\mathbb{P}\mathrm{ic}_{\mathbb{A}^n/\mathbb{C}}$ <u>not</u> representable.

($\exists X$ scheme: $\mathrm{Pic}(X \times \mathbb{A}^1) \neq \mathrm{Pic}(X)$)

(d) (E, p_0) elliptic curve

$$\mathbb{P}\mathrm{ic}^0_{E/\mathbb{C}} \cong E$$

$$\mathcal{L} = \mathcal{O}_{E\times E}(\Delta - p_0 \times E)$$



§2.4 Coarse moduli spaces

⇝ "approximate moduli functor"

<u>Def</u> Given a moduli functor $\mathfrak{h}$, a <u>coarse moduli space</u> is a pair (M, Φ) of a scheme M and nat. transf.

$$\Phi : \mathfrak{h} \longrightarrow h^M$$

Such that

(a) (M, Φ) initial among such pairs:

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "\Phi"] \arrow[dr, "\forall \Phi'"'] & h^M \arrow[d, dashed, "\exists! \Psi"] \\ & h^{M'} \end{tikzcd}$$

(b) Φ induces a bijection

$$\Phi(\mathrm{Spec}\,\mathbb{C}) : \mathfrak{h}(\mathrm{Spec}(\mathbb{C})) \xrightarrow{\sim} h^M(\mathrm{Spec}\,\mathbb{C}) = M(\mathbb{C})$$

on $\mathbb{C}$-points.

<u>Easy exercise</u>

(a) Show: every fine moduli space is a coarse mod. space

(b) Show: assuming prop. a) from def. above, show (M, Φ) is unique up to isomorphism.

<u>Prop</u> Y scheme s.t. $\mathrm{Pic}_{Y/\mathbb{C}}$ is repr. by $\mathbb{P}\mathrm{ic}_{Y/\mathbb{C}}$

$\Rightarrow$ Pic_Y has coarse moduli space $\mathbb{P}\mathrm{ic}_{Y/\mathbb{C}}$.

<u>Idea</u>

map does not ch. by pullback from base

$$\begin{tikzcd} \mathrm{Pic}_Y \arrow[r, "\Phi"] \arrow[dr] & \mathrm{Pic}_{Y/\mathbb{C}} \cong h^{\mathbb{P}\mathrm{ic}_{Y/\mathbb{C}}} \arrow[d] \\ & h^{M'} \end{tikzcd}$$

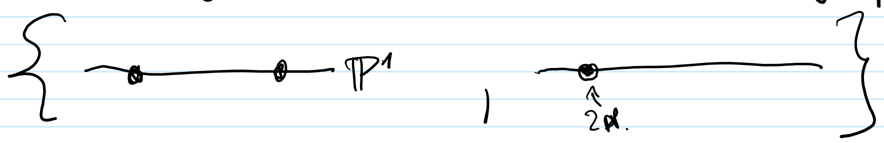
$\square$

Challenge

Recall: $\mathrm{Pic}_{\mathbb{A}^1/\mathbb{C}}$ not representable

Prove or disprove: $\operatorname{Spec}\mathbb{C}$ is coarse mod. space for $\mathrm{Pic}_{\mathbb{A}^1/\mathbb{C}}$.

Exercise Classify two points in $\mathbb{P}^1$ up to proj. equivalence



$\rightsquigarrow$ define mod functor h, show that $(\operatorname{Spec}\mathbb{C}, \Phi: h \to h^{pt})$ satisfies part (a) of Def above, but not part (b)

3| Families of curves and their moduli

3.1| Smooth and nodal curves

Def A complex curve is a one-dim'l variety $C \to \operatorname{Spec}(\mathbb{C})$. In other words: a reduced, separated scheme of fin. type / $\operatorname{Spec}(\mathbb{C})$, st. all irred. components have dim. 1.

Def C complex projective curve:

(a) C smooth, its geometric genus is defined as

$$p_g(C) = h^{1,0}(C) = \dim H^0(C, \Omega^1_C)$$

$\uparrow$ cotangent sheaf. line bdle.

C singular $\Rightarrow$ $\tilde{C}$ normalization smooth curve

$$p_g(C) = p_g(\tilde{C})$$

(b) The arithmetic genus of C is defined as

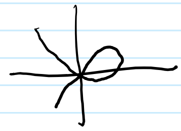
$$p_a(C) = 1 - \chi(C, \mathcal{O}_C) = 1 - \dim H^0(C, \mathcal{O}_C) + \dim H^1(C, \mathcal{O}_C)$$

Prop C smooth, irred. projective $\Rightarrow p_g(C) = p_a(C)$.

Proof $H^0(C, \mathcal{O}_C) = \mathbb{C}$, C irred.

$$\Rightarrow p_a(C) = 1 - 1 + \dim \underbrace{H^1(C, \mathcal{O}_C)}_{\cong H^0(C, \Omega_C^1)^\vee} \overset{\text{Serre dual.}}{=} \dim H^0(C, \Omega_C^1)^\vee = p_g(C) \qquad \square$$

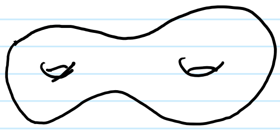
Ex $C \subset \mathbb{P}^2$ nodal cubic curve, e.g.



$$C = \{[X:Y:Z] \in \mathbb{P}^2 : ZY^2 + X^3 - ZX^2 = 0\}$$

Show: $p_a(C) = 1$, $p_g(C) = 0$.

Digression (Riemann surfaces) C smooth, irr. proj.



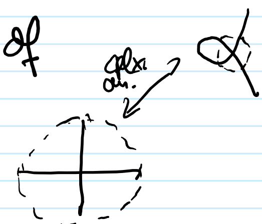
$C(\mathbb{C})$
real surface

$\rightsquigarrow \quad p_g(C) = \underbrace{\#\text{ holes in surface}}_{\text{e.g. } 2}$

Def C complex curve

(a) A closed point $q \in C$ is a node (ordinary double point) if it satisfies one of the equiv. conditions:

- $\exists$ neighbourhood of $q \in C(\mathbb{C})$ which is complex analytic. isomorphic to neighbourhood of origin in
$$\{(x,y) : x \cdot y = 0\} \subset \mathbb{C}^2$$



- the completion $\widehat{\mathcal{O}_{C,q}}$ of the loc. ring at q is isomorphic to $\mathbb{C}[[x,y]]/(x \cdot y)$

(q smooth $\rightsquigarrow \widehat{\mathcal{O}_{C,q}} \cong \mathbb{C}[[x]]$)

(b) C nodal if all $q \in C$ closed pt. are smooth or nodes.

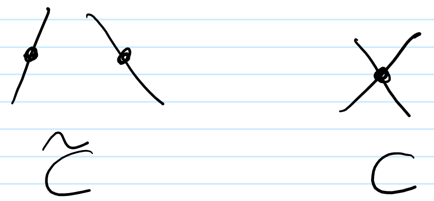
Exerc Show that "nodal cubic curve" is nodal.

C cplx. projective nodal curve $\Rightarrow$ $\nu : \tilde{C} \longrightarrow C$ normalization map.

$\rightsquigarrow$ $\tilde{C}$ cplx. proj. smooth curve,

$\rightsquigarrow$ ν isomorphism over loc. of smooth pts in C

$\rightsquigarrow$ every node $q \in C$ has exactly two preimages q', q''



$\rightsquigarrow$ normalization exact sequence $\quad \nu : \tilde{C} \longrightarrow C$

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \nu_* \mathcal{O}_{\tilde{C}} \longrightarrow \bigoplus_{q \text{ node of } C} \mathbb{C}_q \longrightarrow 0 \qquad (*)$$

$$s \longmapsto (\nu^* s)$$

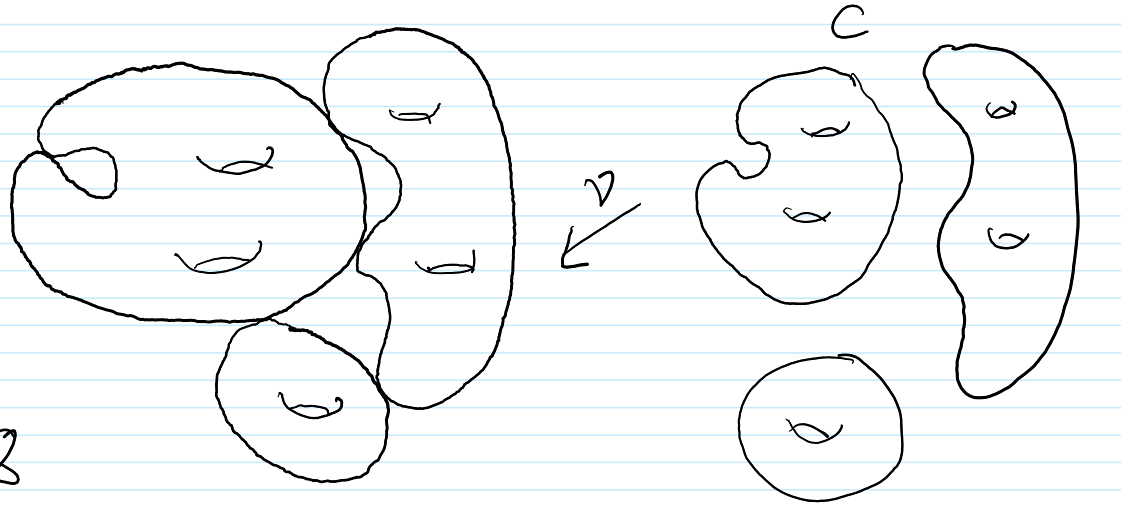
$\uparrow$ skyscraper sheaf at q

$$t \longmapsto \left(t(q') - t(q'')\right)_{\substack{q \text{ node} \\ q', q'' \text{ preimages.}}}$$

Easy exercise Use $(*)$ to show:

$$p_a(C) = p_a(\tilde{C}) + \#\{\text{nodes of } C\}$$

$$p_a(C) = p_g(\tilde{C}) + 1 - \#\{\text{comp. of } \tilde{C}\} + \#\{\text{nodes of } C\}$$



$p_g(C) = 5$

$p_a(C) = 8$