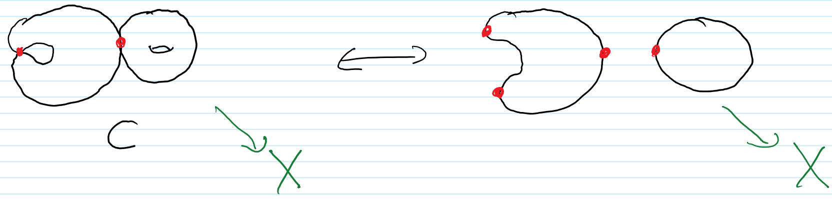
Fact C nodal curve with nodes $q_1, \dots, q_\ell$ $\underset{\text{gluing}}{\overset{\text{normaliz.}}{\longleftrightarrow}}$ $\tilde{C}$ normalization of C together with pairs $\{q_1', q_1''\}, \dots, \{q_\ell', q_\ell''\}$



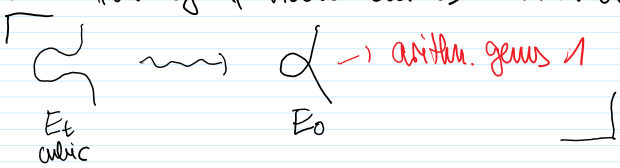
Given scheme X

$$\varphi : C \to X \quad \longleftrightarrow \quad \tilde{\varphi} : \tilde{C} \to X, \quad q_i', q_i'' \mapsto x \in X$$

In next section:

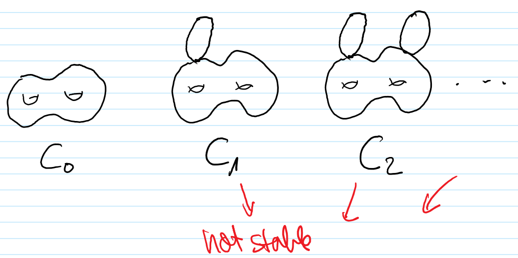
→ define moduli spaces of nodal curves

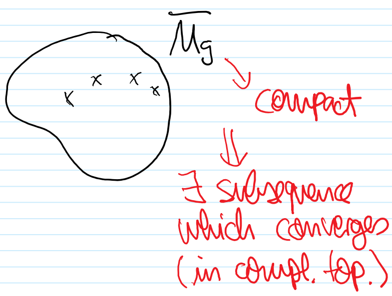
→ in a family of nodal curves: arithmet. genus is (loc.) const.



⇒ $\overline{M}_g$: moduli sp. of nodal curves of arithm. genus g

→ Problem: too many curves of arithm. genus g?





⇝ stable curves

Def A connected, nodal complex proj. curve C is *stable* if its set of automorphisms

$$\mathrm{Aut}(C) = \{ \varphi : C \to C \mid \varphi \text{ isomorph.} \}$$

is finite.

A conn. ... curve C, together with distinct smooth points $p_1, \dots, p_n \in C$. $(C, p_1, \dots, p_n)$ is *stable* if

$$\mathrm{Aut}(C, p_1, \dots, p_n) = \left\{ \varphi : C \to C \;\middle|\; \begin{array}{l} \varphi \text{ isomorphism} \\ \varphi(p_i) = p_i, \ i = 1, \dots, n \end{array} \right\}$$

is finite.

Automorphism groups of smooth curves

Fact C smooth. irred. proj. curve of genus g.

a) For $g=0$ $\quad C \cong \mathbb{P}^1$

$$\mathrm{Aut}(\mathbb{P}^1) = PGL_2(\mathbb{C})$$

$$PGL_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{P}(\mathrm{Mat}_{2\times 2,\mathbb{C}}) \,\middle|\, a\cdot d - b\cdot c \neq 0 \right\} \subset \mathbb{P}(\mathrm{Mat}_{2\times 2,\mathbb{C}}) \cong \mathbb{P}^3$$

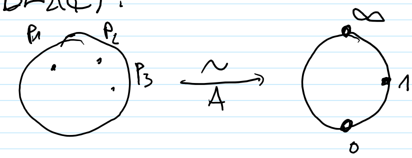
action on $\mathbb{P}^1$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot [X:Y] = [aX+bY : cX+dY]$$

action on $\mathbb{P}^1(\mathbb{C})$ is ${}^{\text{simply}}$ 3-transitive.

Given $p_1, p_2, p_3 \in \mathbb{P}^1(\mathbb{C})$ ${}^{\text{distinct}}$ $\exists!\, A \in PGL_2(\mathbb{C})$:

$$A p_1 = 0 = [0:1]$$
$$A p_2 = 1 = [1:1]$$
$$A p_3 = \infty = [1:0]$$



Equivalently $\quad$ big diagonal $= \{(q_1,q_2,q_3) \mid \exists i,j,\ i\neq j,\ p_i = p_j\}$

$$PGL_2 \longrightarrow (\mathbb{P}^1)^3 \setminus \Delta$$
$$A \longmapsto (A.0,\ A.1,\ A.\infty)$$

isomorph. of alg. varieties.

b) For $g=1$ $\quad$ For $C=E$ of genus 1:

$$\mathrm{Aut}(E) \cong \underset{\text{trafin.}}{E(\mathbb{C})} \rtimes G \leftarrow \text{fin.}$$

$$G \in \{\mathbb{Z}/2\mathbb{Z},\ \mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/6\mathbb{Z}\}$$

$E(\mathbb{C}) \curvearrowright E$ "by translation" $\rightsquigarrow$ E group scheme $\leadsto$ for some neutr. elem. chosen

$$E \times E \xrightarrow{\oplus} E$$
$$(a,b) \longmapsto a \oplus b$$

$\hookrightarrow$ action ${}^{\text{of } E(\mathbb{C})}$ is simply 1-transitive

$\forall\, b,c \in E(\mathbb{C})\ \exists!\, \varphi \in E(\mathbb{C}) \subseteq \mathrm{Aut}(E) \Rightarrow$ maps $E \xrightarrow{\sim} E$, $b \longmapsto a \oplus b$, $\quad a \in E(\mathbb{C})$

$$\varphi(b) = c$$

c) For $g \geq 2$ $\quad \mathrm{Aut}(C)$ is finite

$$\#\mathrm{Aut}(C) \leq 84(g-1)$$

Exercise (easy)

$(C, p_1, \ldots, p_n)$ as above, C smooth ${}^{\text{genus } g}$.

Show: $\mathrm{Aut}(C, p_1, \ldots, p_n)$ is finite

$$\Longleftrightarrow 2g-2+n > 0.$$

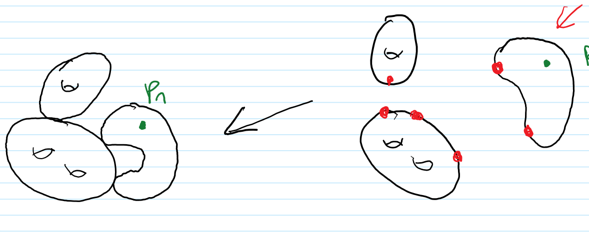
Prop C connected, nodal complex proj. curve, $p_1, \dots, p_n \in C$ distinct smooth points. Then

$(C, p_1, \dots, p_n)$ is stable

$\iff$ every irred. comp. $\tilde{C}_v$ of normalizat. of C satisfies:

- $\tilde{C}_v$ is of genus 0 and contains at least 3 special points (marking p_i, preim. of node of C)
  preim. of   under $\tilde{C} \to C$
- $\tilde{C}_v$ is of genus 1 and cont. at least 1 spec. point
- $\tilde{C}_v$ is of genus at least 2.

Exa



Proof

$$\begin{tikzcd} \tilde{C} \arrow[r, dashed, "\tilde{\varphi}"] \arrow[d, "\nu"'] & \tilde{C} \arrow[d, "\nu"] \\ C \arrow[r, "\varphi"'] & C \end{tikzcd}$$

$\tilde{\varphi}$ autom. of $\tilde{C}$, $\tilde{\varphi}(q_i') = \tilde{\varphi}(q_i'')$ $\in \{$preim. of nodes$\}$, $\{q_i', q_i''\}$ preimage sets of nodes.

$\updownarrow$

φ autom. of C

⇝ use this to show

$$0 \longrightarrow K \longrightarrow \mathrm{Aut}(C, p_1, \dots, p_n) \longrightarrow \underbrace{\mathrm{Sym}\big(\{\tilde{C}_v : \text{comp. of } \tilde{C}\}\big) \times \mathrm{Sym}\Big(\bigcup_{i=1}^{\ell} \{q_i', q_i''\}\Big)}_{\text{finite group.}}$$

fin. sets   set of preim. of nodes

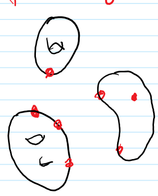
$\mathrm{Aut}(C, p_1, \dots, p_n)$ finite $\iff$ K finite.

$\varphi \in K \longleftrightarrow \tilde{\varphi} : \tilde{C} \longrightarrow \tilde{C}$ automorphism

$\longrightarrow$ leaves all comp. $\tilde{C}_v$ of $\tilde{C}$ invar.

$\longrightarrow$ fixes special points.

$\iff$ ($\tilde{\varphi}_v : \tilde{C}_v \longrightarrow \tilde{C}_v$) fix. spec. pts.



$2g_v - 2 + n_v > 0$

$g_v = 0: \; n_v \geq 3$

$g_v = 1: \; n_v \geq 1$

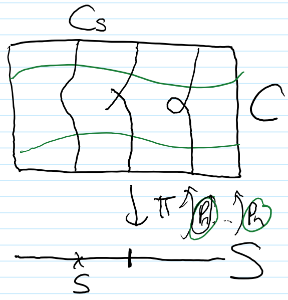
$g_v \geq 2: \; n_v \geq 0$

$\square$

Families of smooth and stable curves

Def Given $g, n \geq 0$, an <u>n-pointed family of smooth (stable) curves</u> of arithm. genus g over a scheme S is a ~~tuple~~

$$\left(\pi : C \to S \,,\, p_1, \ldots, p_n : S \to C \right)$$



where

- π is a smooth (flat) morphism, proper, surjective, finitely presented, such that the fibre $\left(C_s, p_1(s), \ldots, p_n(s)\right)$ over any geom point $s \in S$ is a smooth (stable) proj. connected curve. of arithmetic genus g.
- the morphisms $p_1 \ldots p_n$ are pairwise disjoint sections of π with image in smooth locus of π.

Pullback of families

Given morphism $T \xrightarrow{f} S$, we define pullback of family above as:

$$\begin{tikzcd} T \times_S C = C_T \arrow[r] \arrow[d, shift right, "\pi_T"'] & C \arrow[d, shift right, "\pi"'] \\ T \arrow[r, "f"] \arrow[u, shift right, "p_{i,T}"'] & S \arrow[u, shift right, "p_i"'] \end{tikzcd}$$

$$\Rightarrow f^*\left(\pi : C \to S, p_1 \ldots p_n : S \to C\right) = \left(C_T \xrightarrow{\pi_T} T, p_{1,T}, \ldots, p_{n,T} : T \to C_T\right)$$

Isomorphisms of families of curves

$$\begin{tikzcd} C \arrow[rr, "\varphi", "\sim"'] \arrow[dr, shift left, "\pi"] & & C' \arrow[dl, shift right, "\pi'"'] \\ & S \arrow[ul, bend left, "p_i"] \arrow[ur, bend right, "p_i'"'] & \end{tikzcd}$$

$\varphi : C \to C'$ isomorph. s.t. all diagrams commute.

Rmks

$\rightarrow_\pi$ finitely presented $\rightsquigarrow$ S loc. noetherian, then automatic from π proper.

→ C_s geom. fibres are autom. projective

⇝ why not ask π projective

⇝ not a notion that can be checked on open cover of S

→ π flat, proper loc. fin. presented

⟹ Euler char. of fibres is loc. const.

arithm. genus " " " "

<u>Def</u> Let $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ be the moduli functors

send. a scheme S to the sets

$$\mathcal{M}_{g,n}(S) = \{(\pi: C \to S, p_1 \dots p_n : S \to C) \mid \text{fam. of sm. curves of arithm. genus } g \text{ over } S\} / \text{isom}$$

$$\overline{\mathcal{M}}_{g,n}(S) = \{ \quad - \; - \quad - \quad \mid \; - - \text{ stable } - - - \} / \text{isom.}$$

of n-pointed fam. of smooth / stable curves, up to isom.

Given morph. $f: T \to S$

$$\rightsquigarrow \mathcal{M}_{g,n}(S) \longrightarrow \mathcal{M}_{g,n}(T) \quad , \quad \overline{\mathcal{M}}_{g,n}(S) \longrightarrow \overline{\mathcal{M}}_{g,n}(T)$$

are def. as pullbacks above.

<u>Easy exerc.</u> Check this is well-def. moduli functors.

<u>Thm</u> ([DM69, Kn83]) Let $2g-2+n > 0$.

(a) There exist coarse moduli spaces $M_{g,n}$ and $\overline{M}_{g,n}$ for $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$.

(b) They are normal alg. varieties of dimens. $3g-3+n$ and there exists inclus. $M_{g,n} \subset \overline{M}_{g,n}$ as nonempty open dense subvariety.