Lecture 5

Dienstag, 19. Mai 2020   09:33

Thm ([DM69, Kn83]) Let $2g-2+n>0$.

(a) There exist coarse moduli spaces $M_{g,n}$ and $\overline{M}_{g,n}$ for $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$.

(b) They are normal alg. varieties of dimens. $3g-3+n$ and there exists inclus. $M_{g,n} \subset \overline{M}_{g,n}$ as nonempty open dense subvariety.

(c) $\overline{M}_{g,n}$ is irreducible, projective and has quotient singularities.

Zariski cover: V/G
V affine smooth var.
$G \curvearrowright V$ fin group.

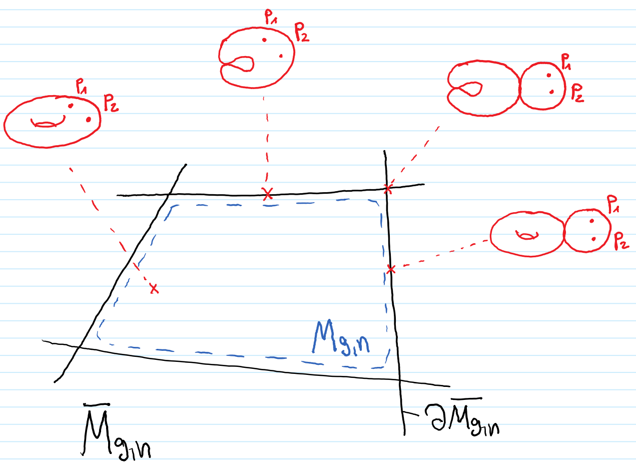
(d) The boundary $\partial\overline{M}_{g,n} = \overline{M}_{g,n} \setminus M_{g,n}$ is the locus of singular curves, and a (Weil) divisor.

(e) The locus $\overline{M}_{g,n}^{\circ} \overset{\text{open}}{\subseteq} \overline{M}_{g,n}$ of curves $(C, p_1, \dots, p_n)$ with trivial aut. group

$$\mathrm{Aut}(C, p_1, \dots, p_n) = \{\mathrm{id}_C\}$$

is a fine moduli space for the functor $\overline{\mathcal{M}}_{g,n}^{\circ}$ of st. curves without nontr. automorphisms. Thus $\exists$ universal family

$$\begin{array}{ccc} \overline{C}_{g,n}^{\circ} & & \\ \pi \downarrow \uparrow p_1, \dots, p_n & & \\ \overline{M}_{g,n}^{\circ} & \subseteq & \overline{M}_{g,n} \end{array}$$



§4 Examples and basic constructions

§4.1 Smooth curves of genus 0

$g=0 \rightsquigarrow 2g-2+n>0 \Leftrightarrow n \geq 3$

$\boxed{n=3}$ $M_{0,3} = ?$ $[1:0]$ $[1:1]$ $[0:1]$

$\boxed{n=3}$ $M_{0,3} = ?$

$$(C, p_1, p_2, p_3) \cong (\mathbb{P}^1, p_1, p_2, p_3) \cong (\mathbb{P}^1, 0, 1, \infty)$$

with $0 = [1:0]$, $1 = [1:1]$, $\infty = [0:1]$.

smooth genus 0 (for C)

$PGL_2(\mathbb{C}) \curvearrowright \mathbb{P}^1(\mathbb{C})$ simply 3-trans.

↑ unique curve in $g=0$, $n=3$ up to isomorph.

$\rightsquigarrow$ expect: $M_{0,3} = pt = \operatorname{Spec} \mathbb{C}$

Prop 1 The variety $M_{0,3} = pt$ is a fine moduli space for $\mathcal{M}_{0,3}$.

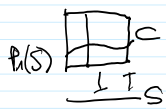
Idea: Start w/ any family

$$\begin{array}{c} C \\ \pi \downarrow \uparrow p_1,p_2,p_3 \\ S \end{array} \;\cong\; \begin{array}{c} S\times\mathbb{P}^1 \\ \pi \downarrow \uparrow 0,1,\infty \\ S \end{array}$$

$(g=0, n=3)$

Prop 2 Let $\pi: C \to S$ be a smooth, proper, surjective, loc. fin. presented morphism of rel. dim 1 w/ geometric fibers isom. to $\mathbb{P}^1$.

- $\to$ If π admits $p_1: S \to C$ section, then $\exists$ rank 2 vector bundle $\mathcal{E}$ on S s.t. $C \cong \mathbb{P}(\mathcal{E})$.
- $\to$ If π has two disjoint sect. $p_1, p_2: S \to C$, then $\mathcal{E} \cong \mathcal{L}_1 \oplus \mathcal{L}_2$, $\mathcal{L}_i$ line bdles.
- $\to$ If π has three disj. sect. $p_1, p_2, p_3: S \to C$ then $\mathcal{E} \cong \mathcal{O} \oplus \mathcal{O} \rightsquigarrow C \cong \mathbb{P}(\mathcal{E}) \cong S \times \mathbb{P}^1$.



Idea

$\mathcal{L} = \mathcal{O}_C(p_1)$

$\pi_* \mathcal{L} = \mathcal{E}$ rank 2

(on fibres: $\mathbb{P}^1$, $\mathcal{L}|_{\mathbb{P}^1} = \mathcal{O}(p_1) \cong \mathcal{O}(1)$) 2-dim'l sect.

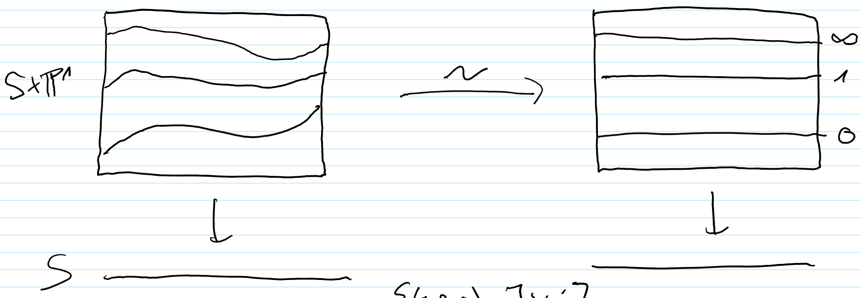
Proof of Prop 1

$$\begin{array}{c} C \\ \pi \downarrow \uparrow\uparrow\uparrow \\ S \end{array} \overset{\text{Prop 2}}{\cong} \begin{array}{c} S\times\mathbb{P}^1 \\ \pi \downarrow \uparrow p_1 \uparrow p_2 \uparrow p_3 \\ S \end{array} \overset{\text{WANT}}{\cong} \begin{array}{c} S\times\mathbb{P}^1 \\ \downarrow \uparrow 0 \uparrow 1 \uparrow \infty \\ S \end{array}$$

$$\mathcal{M}_{03}(S) \cong h^{pt}(S)$$

$$\left\{ \begin{array}{c} S\times\mathbb{P}^1 \\ \downarrow \uparrow 0,1,\infty \\ S \end{array} \right\} \qquad \{ S \to pt \}$$

$\Downarrow$ isom. of funct.



$\Downarrow$

$\{(q_1,q_2,q_3) : \exists i \neq j,\ q_i = q_j\}$

$$A = (p_1, p_2, p_3) : S \longrightarrow (\mathbb{P}^1)^3 \setminus \Delta \cong PGL_2$$

$$\begin{array}{ccc} S \times \mathbb{P}^1 & \longrightarrow & S \times \mathbb{P}^1 \\ (s, q) & \longmapsto & (s, A(s).q) \\ p_1, p_2, p_3 & \longmapsto & 0, 1, \infty \end{array}$$

$\square$

$\underline{n \geq 4}$

$$(C, p_1, \dots, p_n) \cong (\mathbb{P}^1, p_1, \dots, p_n) \xrightarrow[\exists!\, A \in PGL_2]{} \cong (\mathbb{P}^1, 0, 1, \infty, \underset{A.p_4}{p_4'}, \dots, \underset{A.p_n}{p_n'})$$

unique, determine isomorphism class of $(C, p_1, \dots, p_n)$

Prop

$$\Rightarrow M_{0,n} = \left\{ \begin{array}{c} \text{possible} \\ p_4', \dots, p_n' \end{array} \right\} = \left(\mathbb{P}^1 \setminus \{0,1,\infty\} \right)^{n-3} \setminus \Delta$$

$\Delta = \{ (p_i') \mid \exists i \neq j: p_i' = p_j' \}$

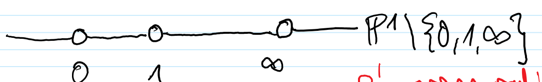
$\rightsquigarrow$ this is true — is a fine moduli space for $M_{0,n}$.

dim $= n-3 = \boxed{3g-3+n}$
irreducible ✓
normal $\rightsquigarrow$ actually smooth
$\Uparrow$
$\overline{M}^{\circ}_{0,n} \cap M_{0,n} = M_{0,n} \Leftarrow \mathrm{Aut}(C, p_1, \dots, p_n) = \{\mathrm{id}_C\}$

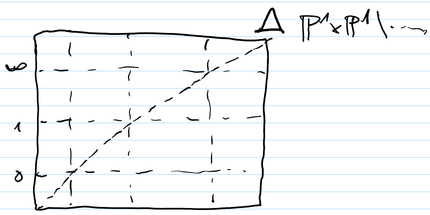
Exerc. Prove this Prop. by imitating proof of Prop. 1 above.
What is the universal family?

Pictures

$M_{0,3}$ $\qquad$ $M_{0,4}$ $\qquad$ $M_{0,5}$

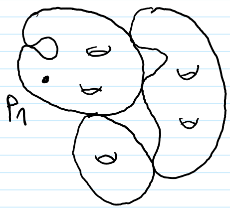


p_4' cross ratio

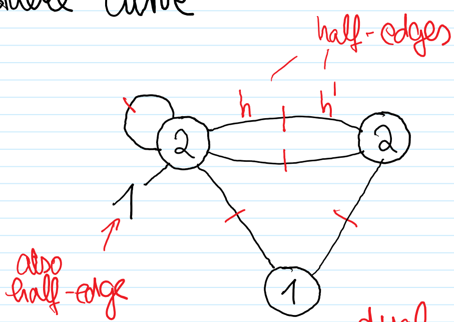


§4.2 Stable graphs and gluing morphisms

$\rightsquigarrow$ Idea Find combinatorial tool to describe "shape" of stable curve



C



Γ_C $\rightsquigarrow$ dual graph of C

Def (Stable graph)

A stable graph Γ is a tuple

$$\Gamma = (V, H, L, g: V \longrightarrow \mathbb{Z}_{\geq 0}, v: H \to V, \iota: H \to H, \ell: L \to \{1, \dots, n\})$$

where

→ $V = V(\Gamma)$ is a finite set (vertices of Γ) and $g: V \longrightarrow \mathbb{Z}_{\geq 0}$ is a map associat. a genus $g(v)$ to each vertex.

→ $H = H(\Gamma)$ is a fin. set (half-edges of Γ). The map $v: H \to V$ associates to each half-edge $h \in H$ a vertex $v(h)$ (vertex incid. to h).

We denote by

$$H(v) = \{h \in H : v(h) = v\} \qquad \text{half-edges at } v$$

$$n(v) = \# H(v)$$

The map $\iota: H \longrightarrow H$ is an involution ($\iota \circ \iota = id_H$).

Thus H decomposes into pairs of half-edg. exchanged by ι and fixed points of ι.

→ The pairs $e = \{h, h'\}$ w/ $\iota(h) = h'$, $\iota(h') = h$, $h \neq h'$, are called the edges $E(\Gamma)$ of Γ

→ The set $L = L(\Gamma) \subseteq H$ is the set of h fixed by ι. (legs of Γ), and $\ell: L \longrightarrow \{1, \dots, n\}$ is a bijective map.

→ The graph Γ is connected, i.e. any two vertices can be connected by a path of edges. ⇝ Mini-Ex: make this precise

→ Stability condition $\forall v \in V$

$$2g(v) - 2 + n(v) > 0$$

An isomorphism $\phi: \Gamma \to \Gamma'$ of stable graphs is a pair of bijections

$$\varphi_V: V \longrightarrow V', \quad \varphi_H: H \longrightarrow H'$$

of their sets of vertices and half-edges st. they are compat. w/ all funct. above

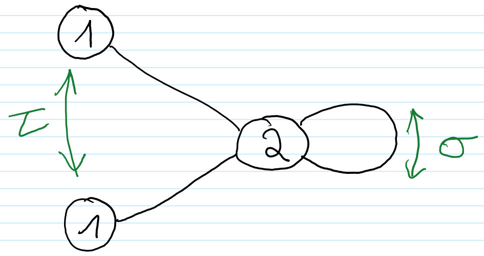
$$g'(\varphi_V(v)) = g(v), \quad v'(\varphi_H(h)) = \varphi_V(v(h)),$$

$$\iota'(\phi_H(h)) = \phi_H(\iota(h)), \quad \ell'(\varphi_H(h)) = \ell(h).$$

$$\mathrm{Aut}(\Gamma) = \{\varphi: \Gamma \to \Gamma \text{ isomorph.}\} \leftarrow \text{form a group under composition}$$

$$g(\Gamma) = \left(\sum_{v \in V(\Gamma)} g(v)\right) + \underbrace{1 + \# E(\Gamma) - \# V(\Gamma)}_{b_1(\Gamma)} \quad 1^{st} \text{ Betti number}$$

$$n(\Gamma) = n = \# L(\Gamma).$$



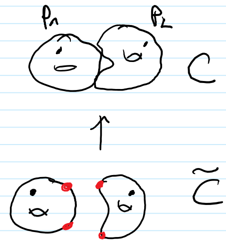
$\tau, \sigma \in \mathrm{Aut}(\Gamma')$ commute

$\mathrm{Aut}(\Gamma') = \langle \tau, \sigma \rangle \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

<u>Exercise</u> Define what the stable graph Γ'_C associated to a stable curve $(C, p_1 \dots p_n) \in \overline{M}_{g,n}$ is.

<u>Easy Exercise</u> Arithm. genus of $C = g(\Gamma'_C)$

<u>Exercise</u> $(C, p_1, \dots, p_n)$ st. curve, $(\tilde{C}_v, (q_h)_{h \in H(v)})_{v \in V(\Gamma')}$ normalizat.



$$0 \to \prod \mathrm{Aut}(\tilde{C}_v, (q_h)_h) \longrightarrow \mathrm{Aut}(C, p_1, \dots, p_n) \longrightarrow \mathrm{Aut}(\Gamma'_C)$$