Lecture 6

Montag, 25. Mai 2020 13:13

Definition 4.8. Given a stable curve $(C, p_1, \ldots, p_n)$ its associated *dual graph* $\Gamma = \Gamma_C$ is the stable graph defined as follows:

- The vertices $v \in V$ of Γ are in one-to-one correspondence to the irreducible components C_v of C (which canonically correspond to the components $\widetilde{C}_v$ of the normalization $\widetilde{C}$).

$$V \cong \{C_v : \text{component of } C\} = \{\widetilde{C}_v : \text{component of } \widetilde{C}\}$$

  The map $g : V \to \mathbb{Z}_{\geq 0}$ sends a vertex v to the genus $g(\widetilde{C}_v)$ of the component in the normalization.

- The half-edges $h \in H$ of Γ are in one-to-one correspondence to the union of the preimages $q', q'' \in \widetilde{C}$ of nodes $q \in C$ under the normalization map $\nu : \widetilde{C} \to C$ and the marked points $p_1, \ldots, p_n \in C$.

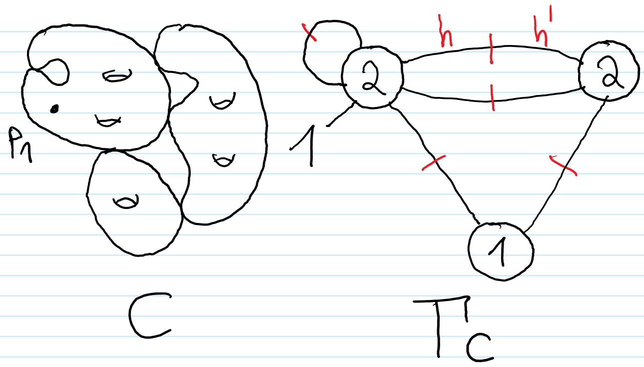
$$H \cong \left(\coprod_{\substack{q \text{ node in } C \\ \nu^{-1}(q)=\{q',q''\}}} \{q', q''\} \right) \sqcup \{p_1, \ldots, p_n\}$$

  The map $v : H \to V$ sends half-edges of the form q', q'' to the vertex v for the component $\widetilde{C}_v$ of the normalization containing them, and the half-edges of the form p_i to the vertex v for the component C_v of C containing them. The involution ι exchanges the preimages of nodes ($\iota(q') = q'', \iota(q'') = q'$) and fixes the marked points ($\iota(p_i) = p_i$).

- The legs $L \subset H$ are precisely the marked points

$$L = \{p_1, \ldots, p_n\}$$

  and the map $\ell : L \to \{1, \ldots, n\}$ sends p_i to i.



Exercise

(a) Show: Γ st. graph, $g(\Gamma) = g$, $n(\Gamma) = n$
$\Rightarrow \#E(\Gamma) \leq 3g - 3 + n$.

(b) Given g, n, show there are only fin many Γ stable gr. $g(\Gamma) = g, n(\Gamma) = n$, up to isomorph.

(c) Show:
$\to$ $\exists!$ graph Γ (g,n) with $\#E(\Gamma) = 0$
$\rightsquigarrow$ trivial stable graph
$\to$ Compute a formula for # of stable gr. Γ w/ $\#E(\Gamma) = 1$ up to isomorphism.

Rmk As g, n grow, there are lots of stable graphs

Exa $g = 1, n = 5$: 15 76 st. graphs up to iso.

$\rightsquigarrow$ can use computer programs

Now $\overline{M}_{g,n}$ decomposes accord. to stable graphs

Prop 1 Let $g, n \geq 0$ w/ $2g - 2 + n > 0$, then for any stable graph Γ the set

$$M^{\Gamma} = \{(C, p_1, \ldots, p_n) \mid \Gamma_C \cong \Gamma\} \subseteq \overline{M}_{g,n}$$

is an irreducible, locally closed, nonempty subset of $\overline{M}_{g,n}$. In particular:

M^{Γ} : strata of $\overline{M}_{g,n}$

$$\overline{M}_{g,n} = \bigsqcup_{\Gamma/iso} M^{\Gamma}$$ ← Stratificat. accord. to stable gr.

and we have

$$\dim M^{\Gamma} = \sum_{v \in V(\Gamma)} 3g(v) - 3 + n(v) = \dim \overline{M}_{g,n} - \# E(\Gamma)$$

codim $= \# E(\Gamma)$ in $\overline{M}_{g,n}$

Idea: nodal curve $\Longleftrightarrow$ normaliz + pairs of pts. which are identified.

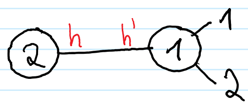
Prop 2: Γ stable gr. of genus g, n legs, then $\exists$ morphism

$$\xi_{\Gamma} : \overline{M}_{\Gamma} = \prod_{v \in V(\Gamma)} \overline{M}_{g(v), n(v)} \longrightarrow \overline{M}_{g,n}$$

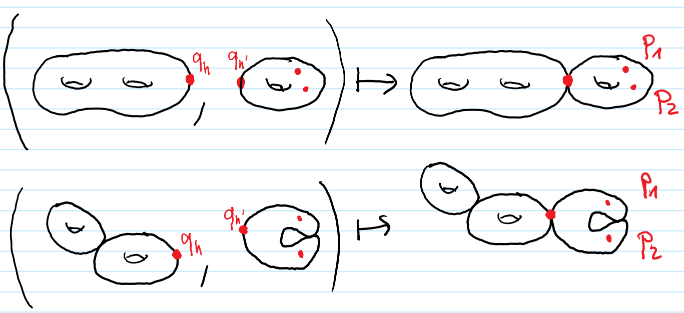
recursive boundary structure of $\overline{M}_{g,n}$

sending a tuple $\left(C_v, (q_h)_{h \in H(v)}\right)_{v \in V(\Gamma)}$ to the curve $(C, p_1, \dots, p_n)$ obtained by gluing pairs $q_h, q_{h'}$ for $\{h, h'\} \in E(\Gamma)$ and setting $p_i \in C$ to be the image of mark. $q_{\ell^{-1}(i)}$ for half-edge $\ell^{-1}(i) \in H(\Gamma)$.
The morphism ξ_{Γ} is finite and its image is the closure

$$\xi_{\Gamma}(\overline{M}_{\Gamma}) = \overline{M^{\Gamma}} \subset \overline{M}_{g,n}.$$

$\Gamma =$ 

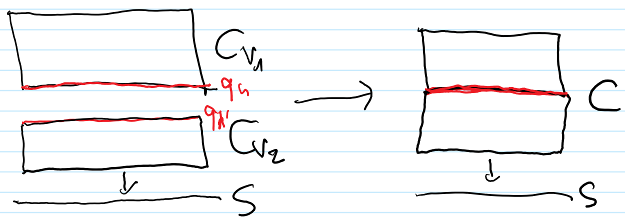
$$\xi_{\Gamma} : \overline{M}_{2,1} \times \overline{M}_{1,3} \longrightarrow \overline{M}_{3,2}$$



<u>Idea of Proof</u> Check: $\overline{M}_\Gamma$ is a coarse moduli space for $\overline{\mathcal{M}}_\Gamma : Sch^{op} \to Set$

$$\overline{\mathcal{M}}_\Gamma(S) = \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)}(S)$$

$$\begin{tikzcd}
(\pi_v : C_v \to S, (q_h : S \to C_v))_v \in \overline{\mathcal{M}}_\Gamma \arrow[d, mapsto] & \overline{\mathcal{M}}_\Gamma \arrow[r] \arrow[d, "\hat{\xi}_\Gamma"'] & h^{\overline{M}_\Gamma} \arrow[d, dashed, "\exists ! \xi_\Gamma"] \\
(C, p_1, \dots, p_n) & \overline{\mathcal{M}}_{g,n} \arrow[r] & h^{\overline{M}_{g,n}}
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
h^S \arrow[r] & h^T \\
S \arrow[r] \arrow[u, Leftrightarrow] & T
\end{tikzcd}$$



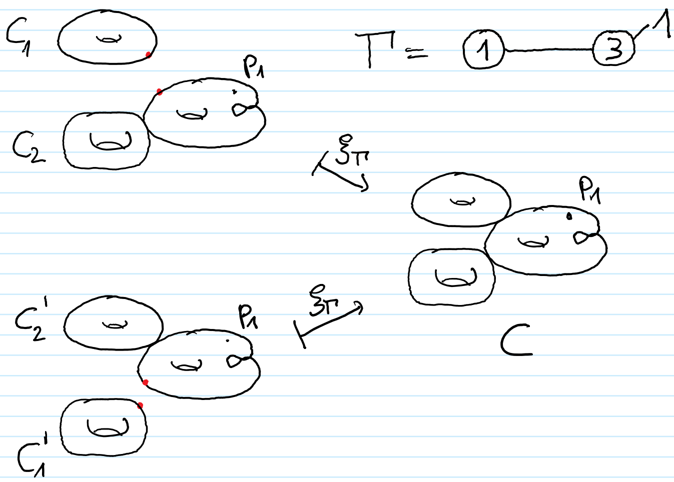
ξ_Γ does correct thing on $\mathbb{C}$-points of $\overline{M}_\Gamma$.

<u>Properties of ξ_Γ</u>

→ ξ_Γ is proper, since its domain is proper, its target is separated } ξ_Γ proper

→ ξ_Γ is quasi-finite (finite fibres)

} → ξ_Γ finite.



Given $(C, p_1, \dots, p_n) \in \xi_\Gamma(\overline{M}_\Gamma)$ what are preimages?

→ $Q \subset \{\text{nodes of } C\}$ ← nodes obtained by gluing

→ Normalize C at pts in Q $\rightsquigarrow C'$
choice of identif. of connect. comp. of C' w/ $V(\Gamma)$
Preim. of nodes w/ $h \in H(\Gamma)$ half-edge

$$\underbrace{}_{\xi_\Gamma^{-1}(\{(C, p_1 \dots p_n)\}) \text{ finite}}$$

→ $\xi_\Gamma(\overline{M}_\Gamma) = \overline{M^\Gamma}$

<u>Start</u> $\xi_\Gamma(M_\Gamma) = M^\Gamma$, $M_\Gamma = \prod_v M_{g(v),n(v)} \subset \overline{M}_\Gamma = \prod_v \overline{M}_{g(v),n(v)}$.

Note $M_\Gamma \subset \overline{M}_\Gamma$ open, dense subset

Note $M_\Gamma \subset \overline{M_\Gamma}$ open, dense, subset (irreducible)
ξ_Γ is proper

$$\xi_\Gamma(\overline{M_\Gamma}) \underset{\uparrow}{\subseteq} \overline{\xi_\Gamma(M_\Gamma)} = \overline{M^\Gamma} \subseteq \underbrace{\xi_\Gamma(\overline{M_\Gamma})}_{\text{closed, irreducible.}} \rightsquigarrow \text{Shows equality.}$$

ξ_Γ continuous

$\square$

Exercise

a) Show that

$$\dim \overline{M_\Gamma} = \sum_{v\in V(\Gamma)} 3g(v)-3+n(v) = \dim \overline{M}_{g,n} - \#E(\Gamma)$$

b) Show that the complement $\partial \overline{M}_{g,n} = \overline{M}_{g,n} \setminus M_{g,n}$ of set of smooth curves is given by the union of $\overline{M^\Gamma}$ for Γ stable graph with exactly one edge.

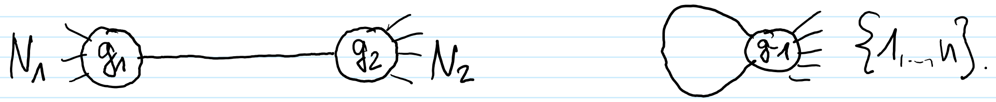
$$\partial \overline{M}_{g,n} = \bigcup_{\Gamma : \#E(\Gamma)=1} \overline{M^\Gamma}.$$

*c) $\forall \Gamma$: $\overline{M^\Gamma} = \bigcup_{\Gamma'} M^{\Gamma'}$ ← describe stable gr. Γ' appearing here.

Concern. b) above

What are stable gr. Γ w/ $\#E(\Gamma)=1$?

$\Gamma_{g_1,N_1} = \Gamma_{g_2,N_2}$ Γ_0



$$g_1+g_2=g$$
$$N_1 \sqcup N_2 = \{1,\dots,n\}$$

$$\Delta_{g_1,N_1} = \overline{M^{\Gamma_{g_1,N_1}}} \qquad \Delta_0 = \overline{M^{\Gamma_0}}.$$

$\rightsquigarrow$ Prop. 1 $\leftsquigarrow$

codim 1

$\Rightarrow$ This proves $\partial \overline{M}_{g,n}$ is Weil divisor.

⇒ This proves $\partial\overline{M}_{g,n}$ is Weil divisor.

<u>Proof of Prop. 1</u> M^Γ loc. closed, nonempty irred, codim $= \# E(\Gamma)$

Saw: $M^\Gamma = \xi_\Gamma(M_\Gamma)$, M_Γ nonempty, irred.

$\rightsquigarrow$ same for M^Γ

<u>Claim</u> $\overline{M^\Gamma} \setminus M^\Gamma = \xi_\Gamma(\overline{M_\Gamma} \setminus M_\Gamma)$ $\quad \subseteq ✓ \quad \supseteq ✓$

proper ← proper, closed in $\overline{M_\Gamma}$, proper

⇒ $M^\Gamma = \overline{M^\Gamma} \setminus (\overline{M^\Gamma} \setminus M^\Gamma)$ open in $\overline{M^\Gamma}$ ⇒ locally closed.

<u>Codim</u> ξ_Γ finite ⇒ $\dim M^\Gamma = \dim \xi_\Gamma(M_\Gamma) \underset{\xi_\Gamma \text{ fin.}}{=} \dim M_\Gamma \underset{\text{exercise}}{=} \dim \overline{M}_{g,n} - \# E(\Gamma)$ □

<u>§4.3</u> Stable curves in genus 0

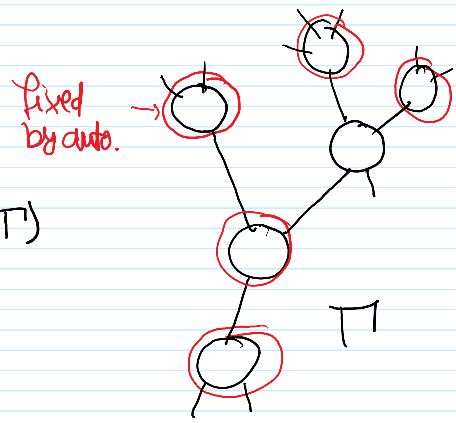
<u>Exercise</u> Γ stable graph of genus 0.

a) Show that the undirected graph with vert. set $V(\Gamma)$ and edges $\{v(h), v(h')\}$ for $\{h,h'\} \in E(\Gamma)$ is a <u>tree</u>.

b) Show $\mathrm{Aut}(\Gamma) = \{\mathrm{id}_\Gamma\}$.

c) Show that any stable curve $(C, p_1, \ldots, p_n)$ of genus 0 has trivial aut. group

$$\mathrm{Aut}(C, p_1, \ldots, p_n) = \{\mathrm{id}_C\}.$$



<u>Corollary</u> For $n \geq 3$, the space $\overline{M}_{0,n}$ is a fine moduli space for the functor $\overline{\mathcal{M}}_{0,n}$ and a smooth, irred. projective variety of dim $n-3$.

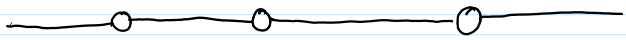
Pf $\overline{M}^\circ_{0,n} = \overline{M}_{0,n}$ + Thm [DM] e) . □

<u>Exa</u> $n=3$

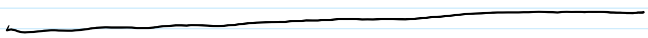
Γ have at most $0 = n-3$ edges ⇒ trivial.

⇒ $\overline{M}_{0,3} = M_{0,3} = \mathrm{pt}$.

Exa $n=4$

$M_{0,4}$  $\mathbb{P}^1 \setminus \{0,1,\infty\}$

$\cap$ open dense

$\overline{M}_{0,4}$  $\mathbb{P}^1$