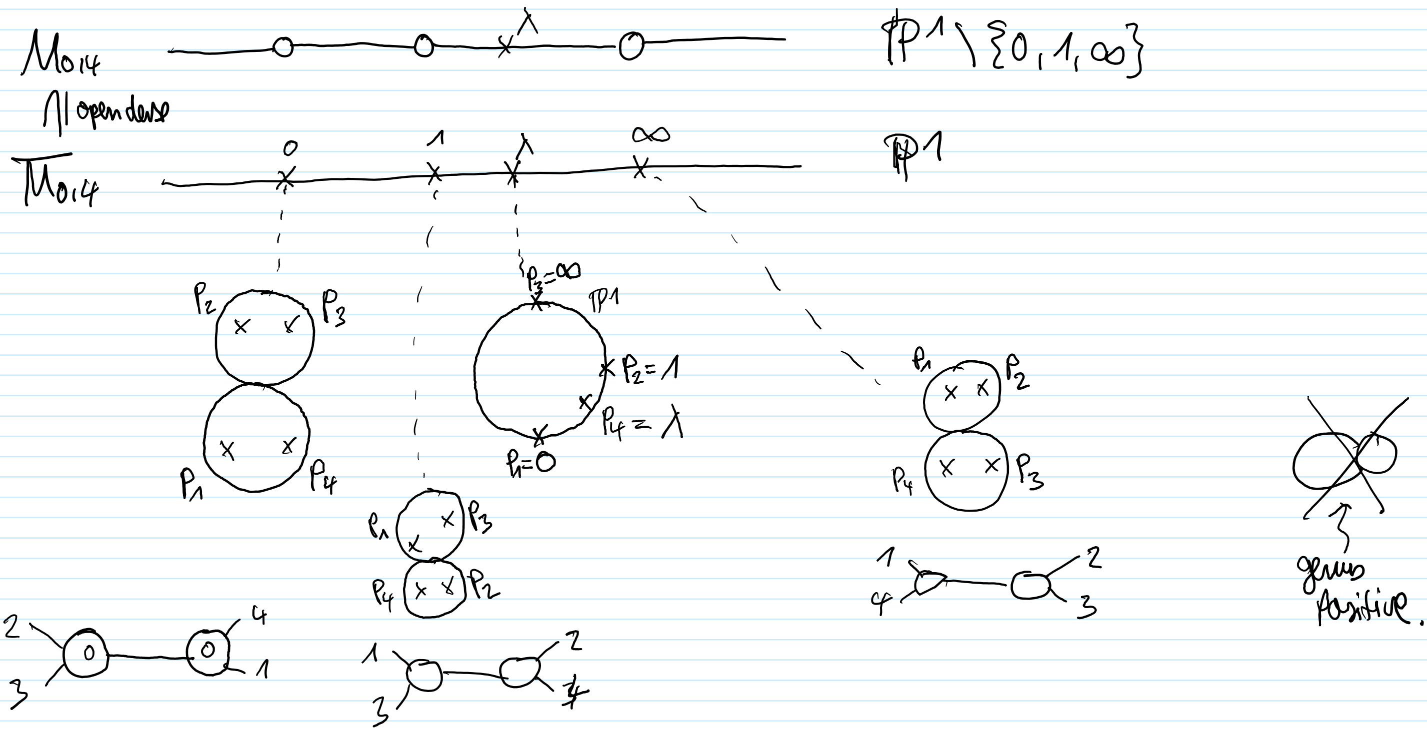
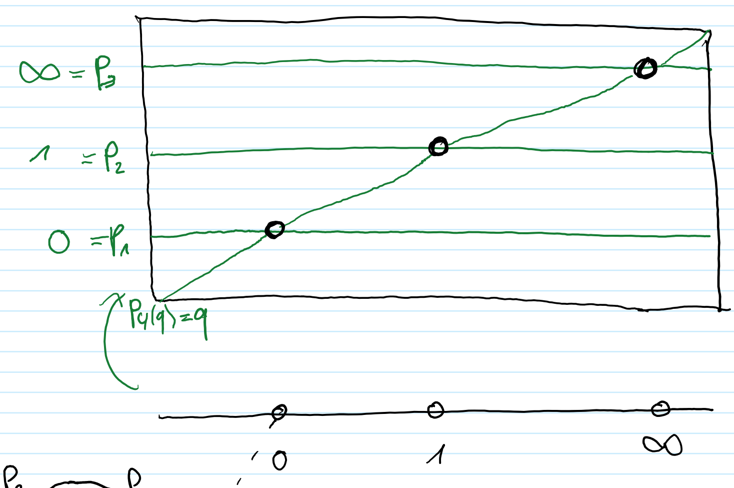
Exa $n=4$



Universal family

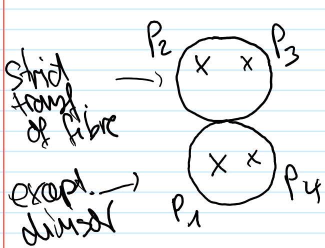
$P_1,\dots,P_4$ strict transforms of sections $0,1,\infty, q\mapsto q$



$\overline{C}_{0,4} = Bl_{\substack{(0,0)\\(1,1)\\(\infty,\infty)}} \mathbb{P}^1\times\mathbb{P}^1$

$\downarrow \; P_1, P_2 \dots P_4$

$\overline{M}_{0,4} \cong \mathbb{P}^1$



$n=5$

Fact $\overline{M}_{0,5} \cong Bl_{\substack{(0,0)\\(1,1)\\(\infty,\infty)}} \mathbb{P}^1\times\mathbb{P}^1 \cong \overline{C}_{0,4}$ universal curve over $\overline{M}_{0,4}$.

Theorem (Knudsen, Keel)

Theorem (Knudsen, Keel)

(a) For $n \geq 3$, we have $\overline{M}_{0,n+1} \cong \overline{C}_{0,n}$ and under this identification, the map

$$\pi : \overline{M}_{0,n+1} \cong \overline{C}_{0,n} \longrightarrow \overline{M}_{0,n}$$

which is the forgetful morphism of marking $n+1$.

$$\pi|_{M_{0,n+1}} : M_{0,n+1} \longrightarrow M_{0,n}$$
$$(C, p_1, \dots, p_n, p_{n+1}) \longmapsto (C, p_1, \dots, p_n)$$

(b) The universal curve $\overline{C}_{0,n} \to \overline{M}_{0,n}$ can be obtained from the projection

$$\overline{M}_{0,n} \times \mathbb{P}^1 \longrightarrow \overline{M}_{0,n} \quad (p_i ?)$$

by an iterated blowup of domain along smooth codim 2 subvarieties.

Remarks

- (a)+(b): allows to construct $\overline{M}_{0,n}$ explicitly by recursion starting from $\overline{M}_{0,3} = pt$.
- $$\begin{array}{ccc} M_{0,n} \times \mathbb{P}^1 & \subseteq & Bl\, \overline{M}_{0,n} \times \mathbb{P}^1 \\ p_n \uparrow \cdots \uparrow p_1 \downarrow & & \downarrow \uparrow p_i \\ M_{0,n} & \subseteq & \overline{M}_{0,n} \end{array}$$
  open dense

  p_i: uniquely determined by their values on dense open $M_{0,n} \subseteq \overline{M}_{0,n}$.

The forgetful morphism and the univ. curve

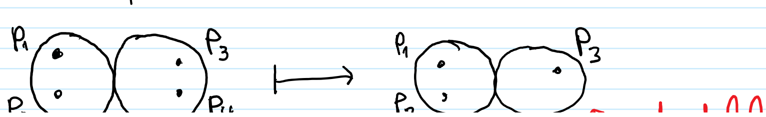
Easy exercise

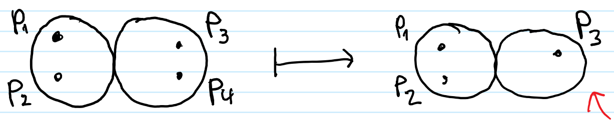
For $2g-2+n > 0$, show there exists a morphism $M_{g,n+1} \to M_{g,n}$ which on complex points is given by

$$M_{g,n+1}(\mathbb{C}) \longrightarrow M_{g,n}(\mathbb{C}), \quad (C, p_1, \dots, p_n, p_{n+1}) \longmapsto (C, p_1, \dots, p_n)$$

Q What goes wrong if we try to do same for $\overline{M}_{g,n+1} \to \overline{M}_{g,n}$?

Hint: $\overline{M}_{0,4} \to \overline{M}_{0,3}$





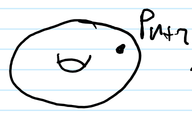
$\nwarrow$ not stable
$\rightsquigarrow$ not allowed.

We will see: $\exists$ extension $\begin{matrix} \overline{M}_{g,n+1} & \xrightarrow{\exists!} & \overline{M}_{g,n} \\ \cup\!\! \mid & & \cup\!\!\mid \\ M_{g,n+1} & \xrightarrow{\checkmark} & M_{g,n} \end{matrix}$ } $\underline{Q}$ What does it do on arbitrary pts of $\overline{M}_{g,n+1}$?

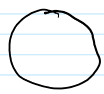
What can go wrong: $(C, p_1, \dots, p_n)$ no longer stable
Let $C_v \subset C$ be the comp. of C containing p_{n+1}.

$\rightsquigarrow$ $\boxed{C_v \text{ is of genus } 0\text{, w/ exactly 3 special points}}$

~~$\rightsquigarrow$ C_v is of genus 1, w/ p_{n+1} only special point~~
$\uparrow$ cannot happen

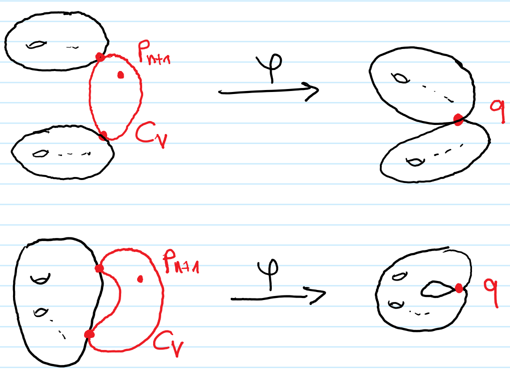


$\rightsquigarrow$ $g=1$, $n=0$, not allowed

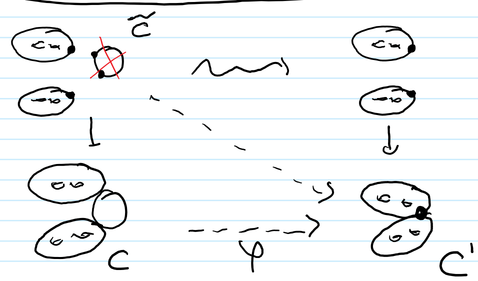


$\uparrow$ disconn. curve

a) $C \underset{\text{stabilization}}{\overset{\text{contraction}}{\rightleftarrows}} C'$



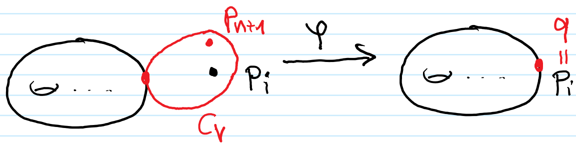
Existence of contraction morphism



$$\overline{M}_{g,n+1}(\mathbb{C}) \longrightarrow \overline{M}_{g,n}(\mathbb{C}) \qquad (\star)$$

$$(C, p_1, \dots, p_n, p_{n+1}) \longmapsto \begin{cases} (C, p_1, \dots, p_n), & \text{if stable} \\ (C', \varphi(p_1), \dots, \varphi(p_n)), & \text{if } (C, p_1, \dots, p_n) \text{ unstable} \end{cases}$$

b)



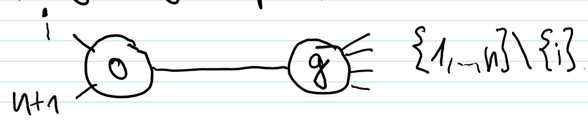
Proposition (Knudsen)

There exists a morphism $\pi: \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ defined on $\mathbb{C}$-points by $(\star)$.
Over the locus $\overline{M}^0_{g,n} \subseteq \overline{M}_{g,n}$ of curves without nontriv. automorphisms, this is the univ. curve

$$\pi\big|_{\pi^{-1}(\overline{M}^0_{g,n})} : \underset{\underset{\overline{C}^0_{g,n}}{\cong}}{\pi^{-1}(\overline{M}^0_{g,n})} \overset{p_1, \dots, p_n}{\rightleftarrows} \overline{M}^0_{g,n}$$

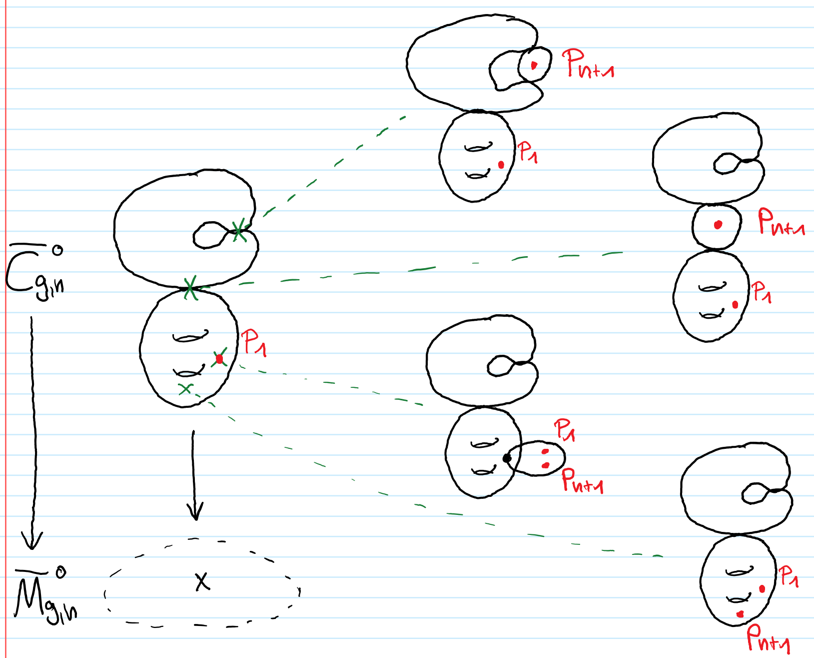
Recall
$g \geq 0$: π univ. curve.

universal sect. $p_i : \overline{M}^{\circ}_{g,n} \longrightarrow \pi^{-1}(\overline{M}^{\circ}_{g,n})$ is given by restriction of gluing map for

$\Gamma_i =$ 

$$\xi_{\Gamma_i} : \underbrace{\overline{M}_{0,3}}_{=pt} \times \overline{M}_{g,n} \longrightarrow \overline{M}_{g,n+1}$$

$$\begin{array}{ccc} \underbrace{\overline{M}_{0,3} \times \overline{M}_{g,n}} & & \overline{M}_{g,n+1} \\ \overline{M}_{g,n} & & \cup\!\vert \\ \cup\!\vert & & \pi^{-1}(\overline{M}^{\circ}_{g,n}) \\ \overline{M}^{\circ}_{g,n} & \xrightarrow{p_i} & \end{array}$$



Sketch of proof

Want to define $\pi : \overline{M}_{g,n+1} \longrightarrow \overline{M}_{g,n}$

$\rightsquigarrow$ functor $\overline{M}_{g,n+1} \cong \overline{\mathcal{C}}_{g,n} \longrightarrow \overline{M}_{g,n}$ $\overline{M}_{g,n}(S) \ni (\pi : C \to S; p_1 \dots p_n)$

natural equiv. of functors

$\overline{\mathcal{C}}_{g,n}(S) = \{ (\pi : C \to S;\ p_1, \dots, p_n, q : S \to C) \mid$ moduli functor

$\to (\pi : C \to S, p_1 \dots p_n : S \to C) \in \overline{M}_{g,n}(S)$

$\to q : S \to C$ <u>any</u> section of $\pi : C \to S$ $\}/_{iso}$

Same trick as in constr. of ξ_{Γ} $\rightsquigarrow$ morphism of coarse mod. spaces.