Lecture 8

Dienstag, 9. Juni 2020 08:51

Moduli spaces of genus 1 curves

$2g-2+n>0 \rightsquigarrow n>0$

Case $n=1$

Prop The moduli functor $\mathcal{M}_{1,1}$ has as coarse moduli space $M_{1,1} \cong \mathbb{A}^1$.

1st lect E_t cubic curves in $\mathbb{P}^2$ $(t \in \mathbb{A}^1 \setminus \{0,1\})$

$\rightsquigarrow$ classified by j-invariant $j : \mathbb{A}^1 \setminus \{0,1\} \to \mathbb{A}^1$, $t \mapsto j(t)$

$(E,p) \xrightarrow{\sim} (E,q) \quad \forall p,q \in E$

background.

Idea Start with (E,p), E smooth genus 1, $p \in E$

$\mathcal{L} = \mathcal{O}_E(3p)$ line bdle on E

$h^0(\mathcal{L}) = 3 \qquad h^1(\mathcal{L}) = 0$

$(E, \mathcal{L}, \underbrace{s_0, s_1, s_2}_{\text{don't vanish simultan.}})$

$\rightsquigarrow E \longrightarrow \mathbb{P}^2$ def. $[s_0 : s_1 : s_2]$

Riemann-Roch $\chi(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L}) = \deg(\mathcal{L}) + 1 - g(E)$

$= 3 + 1 - 1 = 3$

$h^1(\mathcal{L}) = \dim H^1(E, \mathcal{L}) \overset{\text{Serre-Dual.}}{=\!=} \dim H^0(E, \underbrace{\mathcal{L}^\vee \otimes \omega_E}_{\deg = -3+0=-3})^\vee = 0$

$\omega_E \cong \Omega^1_E$ degree $2g(E)-2=0$

If $\mathcal{L}^\vee \otimes \omega_E$ had nonzero sect. $s \rightsquigarrow \mathrm{div}(s)$ has deg -3

$\rightsquigarrow$ **Prop** $\mathcal{L}$ on C linebdle on smooth curve, $\deg(\mathcal{L}) > 2g-2$, $g = g(C)$

$\Rightarrow h^1(\mathcal{L}) = 0$

Summary Get $E \hookrightarrow \mathbb{P}^2$ via $(\mathcal{L}, s_1, s_2, s_3)$

embedding as cubic curve.

$\rightsquigarrow$ compute j-invariant of E (indep. of choices)

$$M_{1,1} \xrightarrow{\sim} \mathbb{A}^1$$
$$(E,p) \longmapsto j(E)$$

full proof: have to do this in families.

Prop $\overline{M}_{1,1} \cong \mathbb{P}^1$ | **Proof** $\overline{M}_{1,1}$ $\underbrace{\text{dim } 1, \text{ normal variety}}$

Prop $\overline{M}_{1,1} \cong \mathbb{P}^1$

$\cup$

$\mathbb{A}^1 \cong M_{1,1}$

$\dim \overline{M}_{1,1} = 3g-3+n$
$= 1$

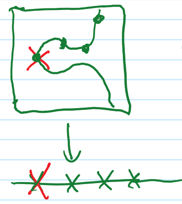
Proof $\overline{M}_{1,1}$ $\underbrace{\text{dim } 1, \text{ normal variety}}$ $\Rightarrow$ smooth

smooth, proj.
irred. + cont. $\mathbb{A}^1$ as open subset $\Big\} \Rightarrow \overline{M}_{1,1} \cong \mathbb{P}^1$ $\square$

What is curve corr. to $\infty \in \mathbb{P}^1 \cong \overline{M}_{1,1}$?

$\overline{M}_{1,1} \cong \overline{M}_{0,4}$ $\rightsquigarrow$ coincidence?

$\cong \overline{M}_{0,4}/S_3$



E cubic
ell. curve

$E \longleftrightarrow \left(\begin{smallmatrix}4 \text{ pts} \\ \text{in } \mathbb{P}^1\end{smallmatrix}\right) / \text{ordering}, \text{Aut}(\mathbb{P}^1)$

$4 \text{ pts}/\text{Aut}(\mathbb{P}^1) \cong M_{0,4}$
$\cong \mathbb{P}^1 \setminus 3\text{pts}$

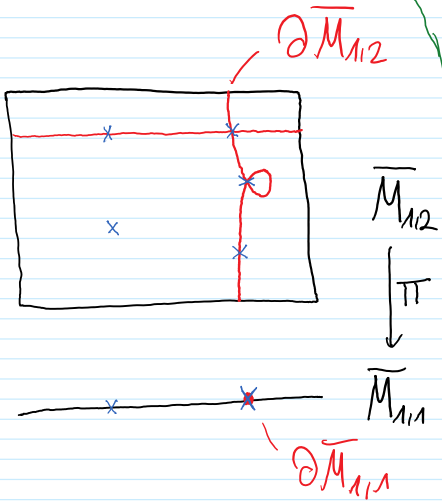
$\rightsquigarrow S_4 \curvearrowright \overline{M}_{0,4}$

$\rightsquigarrow$ quot. isom.
also to $\mathbb{P}^1$

Case n=2

Exercise

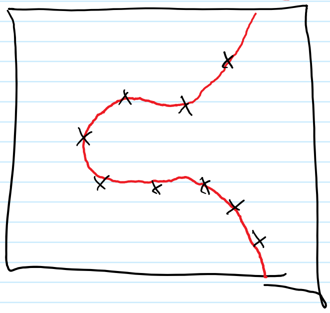
Draw all curves
corresp. to crosses $\times$
in Picture,
and corresp. stable graph.



A fun construct. for n = 9

Exerc.

$\exists!$ cubic E_p through $p_1, \dots, p_9$



9 gen. pts in $\mathbb{P}^2$

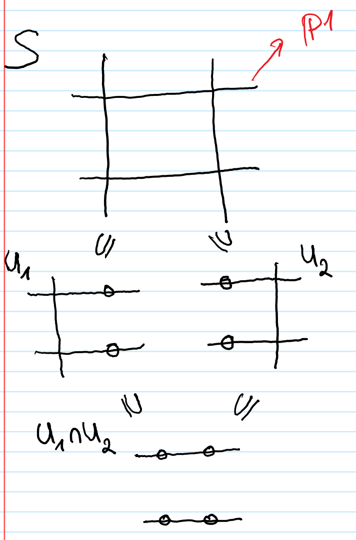
$\rightsquigarrow (\mathbb{P}^2)^9 \dashrightarrow \overline{M}_{1,9}$ rat'l map.

$(p_1, \dots, p_9) \longmapsto (E_p, p_1, \dots, p_9)$ dominant.

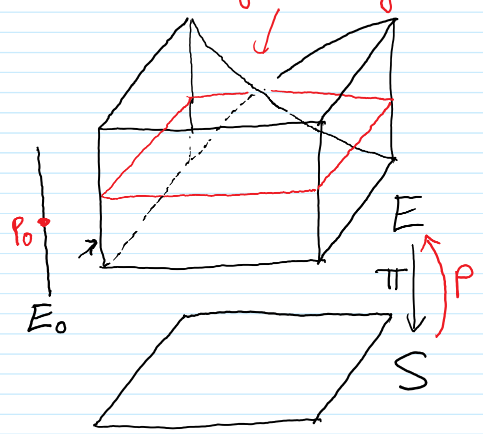
$\Rightarrow \overline{M}_{1,9}$ is unirational (dom. rat'l map from proj. space)

Why is $\overline{M}_{1,1}$ not a fine moduli space

Why is $M_{1,1}$ not a fine moduli space



glue along $\sigma : E_0 \xrightarrow{\sim} E_0$



<u>Exerc</u> family is
→ Zar. loc. trivial
→ not globally triv.

$$\begin{array}{ccc} S & \longrightarrow & \overline{M}_{1,1} \\ \cup \quad \cup & & \uparrow \\ U_1 \quad U_2 & \longrightarrow & (E_0, p_0) = pt \end{array}$$

⇝ contrad. to (E,p) being a pullback from $\overline{M}_{1,1}$ $\quad \overline{C}_{1,1} \downarrow \overline{M}_{1,1}$

□

Moduli stacks of curves

Schemes ⇝ Stacks, Algebraic Stacks

What do we want from new theory?

→ any scheme S can be interpret. as stack
→ can do algebraic geometry w/ stacks $\mathcal{X}$
($\mathcal{X}$ smooth, $\mathcal{X} \to \mathcal{Y}$ proper, ...)
→ ∃ algebraic stack $\mathcal{M}_{g,n}$ serving as a "moduli stack of stable curves"
($\{$morphisms $S \to \mathcal{M}_{g,n}\} \cong$ fam. of stable curves$/S$)
→ algebraic stack $\overline{\mathcal{M}}_{g,n}$ has nicer prop. than $\overline{M}_{g,n}$
($\overline{\mathcal{M}}_{g,n}$ smooth, proper, has univ. family, ...)

An outline of the theory of algebraic stacks

<u>We saw above</u>: family of curves over S was <u>not</u> determined by restrict. to a cover $\overset{S = U_1 \cup U_2}{}$ of S
⌈define gluing data on overlap $U_1 \cap U_2$ ⇝ then uniquely determ.⌋

$S \to M_{g,n}$ (any scheme) <u>is</u> determined on open cover
⌈no nontrivial gluing⌋

⇝ Central idea of stacks $\mathcal{X}$: Morph. to $\mathcal{X}$ det. by
↗ restrict. to open cover of domain
↘ gluing data

⇝ Central idea of stacks $\mathfrak{X}$: Morph. to $\mathfrak{X}$ det. by ↗ ... domain
↘ gluing data

$S = U_1 \cup U_2$

Category theory
f_1, f_2 functors

→ restr. to open cover: $U_1 \xrightarrow{f_1} \mathfrak{X}$
$U_2 \xrightarrow{f_2} \mathfrak{X}$

→ gluing data: $f_1|_{U_1 \cap U_2} \underset{\sim}{\xrightarrow{\varphi_{12}}} f_2|_{U_1 \cap U_2}$

φ_{12} natural equivalence
⇓
right language to describe our stacks.

⇒ Algebraic stacks are categories (+ add. data)

Moduli stack $\overline{\mathcal{M}}_{g,n}$

↳ Ob: $(\pi: C \to S\,;\ p_1, \ldots, p_n : S \to C)$
fam. of stable gen. g, n-marked curves over some scheme S

omit $p_1 \ldots p_n$

↳ Mor: $\mathrm{Mor}\left(\begin{matrix} C' \\ \downarrow \pi' \\ S' \end{matrix}, \begin{matrix} C \\ \downarrow \pi \\ S \end{matrix} \right) = \left\{ \begin{tikzcd} C' \arrow[r, "\hat{f}"] \arrow[d, "\pi'"] & C \arrow[d, "\pi"] \\ S' \arrow[r, "f"] & S \end{tikzcd} (*) \;\middle|\; \begin{matrix} (f, \hat{f}) \text{ st. } (*) \\ \text{is a fibre diagram} \\ \text{(Pullback of fam. of curves} \end{matrix} \right\}$

There is a functor $\mathcal{F}: \overline{\mathcal{M}}_{g,n} \longrightarrow Sch_{\mathbb{C}}$ ← Schemes over $\mathbb{C}$

$(\pi: C \to S) \longmapsto S$

$\left(\begin{tikzcd} C' \arrow[r, "\hat{f}"] \arrow[d] & C \arrow[d] \\ S' \arrow[r, "f"] & S \end{tikzcd} \right) \longmapsto (S' \xrightarrow{f} S)$

has nice properties:

→ For $S \in Ob(Sch_{\mathbb{C}})$ the preimage / fibre $\mathcal{F}^{-1}(S)$ is subcat. of $\overline{\mathcal{M}}_{g,n}$

→ Ob: Families of curves over S

→ Mor: Def Morphisms $(\hat{f}, f)$ mapping to $f = id_S$

$$\begin{tikzcd} C' \arrow[r, "\hat{f}"] \arrow[d] & C \arrow[d] \\ S \arrow[r, equal, "id"] & S \end{tikzcd}$$

$\hat{f}$ makes this a fibre diagram
⇕
$\hat{f}$ isom. of famil. of curves.

this is reason why we defined Mor in $\overline{\mathcal{M}}_{g,n}$ as fibre diagr.

→ "Pullbacks exist"

Given $(\pi : C \to S)$ and $f : S' \to S$

$\exists$ pullback

$$\begin{tikzcd} C' \arrow[r] \arrow[d] & C \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

} $\exists$ morph. $(\hat{f}, f)$ mapping to given $f : S' \to S$.

How can we see schemes M as categories?

$\rightsquigarrow$ Sch_M schemes over M

Ob : $X \to M$

Mor :
$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & M & \end{tikzcd}$$

$\rightsquigarrow$ $F : Sch_M \longrightarrow Sch_{\mathbb{C}}$

$(X \to M) \longmapsto X$

$\left(\begin{tikzcd}[sep=small] X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & M & \end{tikzcd}\right) \longmapsto X \to Y$

} $F^{-1}(X)$

Ob : $Mor(X, M)$

Mor :
$$\begin{tikzcd} X \arrow[rr, "id_X"] \arrow[dr] & & X \arrow[dl] \\ & M & \end{tikzcd}$$

<u>Def</u> $(\mathcal{M}, F : \mathcal{M} \to Sch_{\mathbb{C}})$ categories fibred in groupoids

inf.

$F^{-1}(S)$ groupoid = (category in which all morphisms are isomorphisms)

$\rightsquigarrow$ stacks are particular examples of cat. fibred in groupoids.

Upgraded Yoneda embedding

$$\begin{tikzcd}[column sep=large] & & \text{categories fibred in groupoids} \arrow[d] \\ Sch_{\mathbb{C}} \arrow[r, hook, "M \mapsto h^M"'] \arrow[urr, "M \mapsto Sch_M"] & & \text{moduli functors} \end{tikzcd}$$

$(\mathcal{M}, F : \mathcal{M} \to Sch_{\mathbb{C}})$

$\downarrow$

$(S \mapsto F^{-1}(S)/\text{isom.})$

---- to be continued next week ----

Here is a list of resources, ordered in increasing comprehensiveness, which you can use to learn more about stacks:

- the paper "Stacks for Everybody" [Fan01] by Barbara Fantechi (11 pages, a few hours to work through, highly recommended),
- the course on the topic given by Prof. Georg Oberdieck in the Winter semester 2020 (one semester, also highly recommended),
- the book "Algebraic Stacks" (in preparation, by Behrend, Conrad, Edidin, Fantechi, Fulton, Göttsche und Kresch), found on the website of an old course by Andrew Kresch (220 pages, a few months, a great resource for self-study),
- the Stacks project [Sta13] (about 7000 pages, several years of intense study, great to look up results and particular topics, highly non-recommended to read from beginning to end).