Lecture 9

Dienstag, 16. Juni 2020 09:28

$\rightsquigarrow$ categories fibred in groupoids

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "f"] \arrow[dr, "F"'] & & \mathcal{M}' \arrow[dl, "F'"] \\ & Sch_{\mathbb{C}} & \end{tikzcd}$$

f functors s.t. diagram commutes.

Exa What are morph. $\underbrace{Sch_S}_{\cong S} \longrightarrow \overline{\mathcal{M}}_{g,n}$

$(X \xrightarrow{g} S) \longmapsto g^*(\pi: C \to S, \dots)$

$(S \xrightarrow{id} S) \longmapsto \underbrace{(\pi: C \to S, p_1, \dots, p_n: S \to C)}$

$$\begin{tikzcd} Sch_S \arrow[rr] \arrow[dr] & & \overline{\mathcal{M}}_{g,n} \arrow[dl] \\ & Sch_{\mathbb{C}} & \end{tikzcd}$$

$\Longleftrightarrow$ fam. of curves over S

In general, to do algebraic geom, cat. f. in groupoids are too general.
$\rightsquigarrow$ (algebraic) stacks : cfg w/ special prop.

<u>$\overline{\mathcal{M}}_{g,n}$ is a stack</u> $\rightsquigarrow$ $\overline{\mathcal{M}}_{g,n}$ has "sheaf-like property"

S scheme, $\{U_i \longrightarrow S\}_i$ étale cover of S

$(\pi: C_i \longrightarrow U_i)_i$ families of curves over U_i $\rightsquigarrow$ $U_i \xrightarrow{f_i} \overline{\mathcal{M}}_{g,n}$

s.t. on overlaps $U_i \cap U_j = U_i \times_S U_j$ we are given isomorphisms

$$\begin{tikzcd} \varphi_{ij}: C_i|_{U_i \cap U_j} \arrow[rr, "\sim"] \arrow[dr] & & C_j|_{U_i \cap U_j} \arrow[dl] \\ & U_i \cap U_j & \end{tikzcd}$$

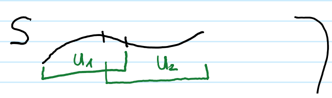
$\rightsquigarrow$ $U_i \cap U_j \underset{f_j}{\overset{f_i}{\rightrightarrows}} \overline{\mathcal{M}}_{g,n}$

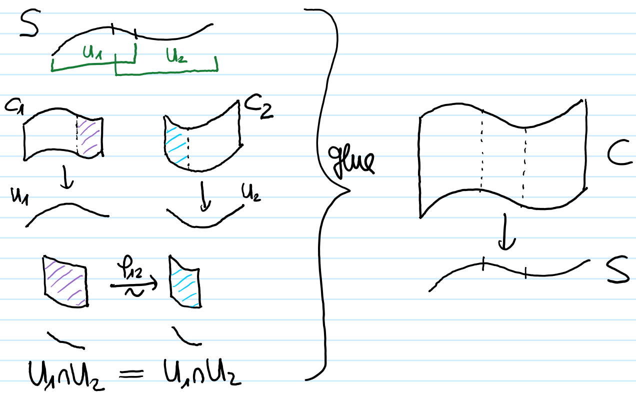
natural equiv. $f_i \xrightarrow{\sim} f_j$

satisfying cocycle condition $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $U_i \cap U_j \cap U_k$

$\Rightarrow$ $\exists$ gluing $(\pi: C \longrightarrow S)$ w/ $C|_{U_i} = C_i$
unique up to unique isom.

$\rightsquigarrow$ $S \xrightarrow{f} \overline{\mathcal{M}}_{g,n}$





Given stacks $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ and

$$\begin{tikzcd} \exists\, \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \arrow[r, dashed] \arrow[d, dashed] & \mathcal{Z} \arrow[d] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

We say that $f: \mathcal{X} \to \mathcal{Y}$ is <u>representable</u> if for every scheme U and morphism $U \longrightarrow \mathcal{Y}$ $(Sch_U \longrightarrow \mathcal{Y})$

$$\begin{tikzcd} Sch_S \cong \mathcal{X} \times_{\mathcal{Y}} U \arrow[r, "f'"] \arrow[d] & U \arrow[d] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

S scheme

"use $U \longrightarrow \mathcal{Y}$ as charts of $\mathcal{Y}$ by schemes"

Now given any property $\mathcal{P}$ of morphisms of schemes invariant under pullback (e.g. "f is smooth")

we say that represent. morph. $f: \mathcal{X} \to \mathcal{Y}$ has $\mathcal{P}$ if and only if $f': \mathcal{X} \times_{\mathcal{Y}} U \longrightarrow U$ has $\mathcal{P}$ for <u>all</u> $U \longrightarrow \mathcal{Y}$.

<u>$\overline{\mathcal{M}}_{g,n}$ is an algebraic stack</u>

$\mathcal{X}$ Being an algebraic stack means that $\exists\; U \xrightarrow{\pi} \mathcal{X}$ such that U scheme, π is representable, smooth, surjective.

"nice cover of $\mathcal{X}$ by a scheme"

To check this for $\overline{\mathcal{M}}_{g,n}$:

$U \longrightarrow \overline{\mathcal{M}}_{g,n} \iff \begin{array}{c} C \\ \downarrow \\ U \end{array}$ family of stable curves

e.g. "U surjective" $\iff$ every stable curve C occurs as a fibre of morphism $\mathcal{C} \to U$

We saw an example of this: $(g,n)=(1,1)$

$\rightsquigarrow \mathcal{M}_{1,1}$ ~~smooth~~ stable ~~ellipt.~~ genus 1 curves

$$\begin{tikzcd} E \arrow[d, "\pi"'] & \{([X:Y:Z],t)\in\mathbb{P}^2\times\mathbb{A}^1 \mid Y^2Z+X(X-Z)(X-tZ)=0\} \\ \mathbb{A}^1 \ni t \arrow[u, bend right, "P_1"'] & \end{tikzcd}$$

$$P_1(t)=([0:1:0],t)$$

$\rightsquigarrow$ fam. of curves over $\mathbb{A}^1$

$\rightsquigarrow \mathbb{A}^1 \longrightarrow \overline{\mathcal{M}}_{1,1}$, $\mathbb{A}^1 = U$

Given an alg. stack $\mathcal{X}$ w/ rep. smooth surj. cover $U\to\mathcal{X}$ and property Q of schemes which can be checked on a smooth cover (e.g. being smooth), then we say $\mathcal{X}$ has Q $\Leftrightarrow$ U has Q

$\rightsquigarrow \overline{\mathcal{M}}_{g,n}$ is smooth if we can find $U\to\overline{\mathcal{M}}_{g,n}$ w/ U smooth

In fact, we can find $U\longrightarrow\overline{\mathcal{M}}_{g,n}$ étale $\overset{\text{Def.}}{\Longrightarrow}$ $\overline{\mathcal{M}}_{g,n}$ is a Deligne-Mumford stack

$\Updownarrow$ nontrivial, stack separat., char $=0$

$\forall\, x\to\overline{\mathcal{M}}_{g,n}$ geom. point: $\mathrm{Aut}(x)$ finite

DM stacks are "very close to being" schemes

$\exists$ morphism

$$\overline{\mathcal{M}}_{g,n} \longrightarrow \overline{M}_{g,n}$$

stack — coarse moduli space

- bijection on geom. points
- proper morphism
- $$\begin{tikzcd} \overline{\mathcal{M}}_{g,n} \arrow[r] \arrow[d] & M \\ \overline{M}_{g,n} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$ any scheme

coarse moduli space

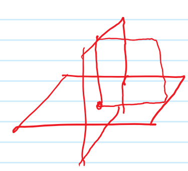
Upgrades of previous results

Thm ([DM]) Let g,n with $2g-2+n>0$.

a) The categories fibred in groupoids $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ are algebraic Deligne–Mumford stacks.

b) They are irreducible, proper and <u>smooth</u> of dimension $3g-3+n$ and $\mathcal{M}_{g,n}\subseteq\overline{\mathcal{M}}_{g,n}$ open substack.

c) The boundary $\partial\overline{\mathcal{M}}_{g,n}=\overline{\mathcal{M}}_{g,n}\setminus\mathcal{M}_{g,n}$ is an effective Cartier divisor and even normal crossings!



d) The forgetful morphism $\pi:\overline{\mathcal{M}}_{g,n+1}\longrightarrow\overline{\mathcal{M}}_{g,n}$ is the univ. curve over $\overline{\mathcal{M}}_{g,n}$. In particular, it is representable, proper flat of rel. dim 1.

e) Given a stable graph Γ of gen. g w/ n legs, the gluing morphism

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma = \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

is representable, finite and a local complete intersection. factored (smooth morph) $\circ$ (closed embedd)

It has generic degree $\# \mathrm{Aut}(\Gamma)$ onto its image $\overline{\mathcal{M}}^\Gamma$.

Eg. For forgetful morphism: write down functor $\overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$

$$(\pi : C \to S, p_1, \ldots, p_{n+1}) \longmapsto (\pi : C' \to S, p_1', \ldots, p_n')$$

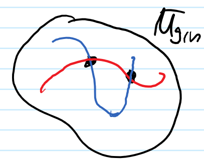
$\Rightarrow$ gives you morph. of stacks.

Intersection theory on moduli of curves

$\rightsquigarrow$ Study topological invariants of $\overline{\mathcal{M}}_{g,n}$

look at $\overline{M}_{g,n}(\mathbb{C})$ together with complex topology. $\overline{M}_{0,4}(\mathbb{C}) \overset{?}{=} \mathbb{CP}^1 \cong S^2$

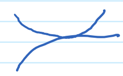
$$H^*(\overline{\mathcal{M}}_{g,n}) = H^*(\overline{M}_{g,n}(\mathbb{C}), \mathbb{Q}) \quad \text{singular cohomology.}$$



Later $S \subset \overline{\mathcal{M}}_{g,n}$ closed alg. subset. of cplx. codim c

$\rightsquigarrow [S] \in H^{2c}(\overline{\mathcal{M}}_{g,n})$

Given $S' \subset \overline{\mathcal{M}}_{g,n}$ second closed alg. subset: $S \cap S'$ transversally



$$[S] \cup [S'] = [S \cap S'] \in H^{2(c+c')}(\overline{\mathcal{M}}_{g,n})$$

$\rightsquigarrow$ origin of the word "intersection theory".

In $\overline{\mathcal{M}}_{g,n}$: closed sets $\overline{\mathcal{M}}^\Gamma$ for Γ stable graph.

closure of strata of $\overline{\mathcal{M}}_{g,n}$

Intersections of strata of $\overline{\mathcal{M}}_{g,n}$

Saw before

$$\overline{\mathcal{M}}_{g,n} = \coprod_{\Gamma \text{ st. graph/isom}} \mathcal{M}^\Gamma$$

loc. closed subsets.

$\mathcal{M}^\Gamma = \{(C, p_1, \ldots, p_n) \mid \Gamma_C \cong \Gamma\}$

$\overline{\mathcal{M}}^\Gamma$ closure of $\mathcal{M}^\Gamma$ (closed alg. subset)

$\xi_\Gamma : \overline{M}_\Gamma = \prod_{v\in V(\Gamma)} \overline{M}_{g(v),n(v)} \longrightarrow \overline{M}_{g,n}$ gluing map.

Satisfied: $\xi_\Gamma(\overline{M}_\Gamma) = \overline{M}^\Gamma$.

Questions

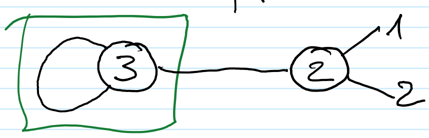
a) Given Γ, what are stable graphs Γ' of curves $(C,p_1,\dots,p_n) \in \overline{M}^\Gamma$?

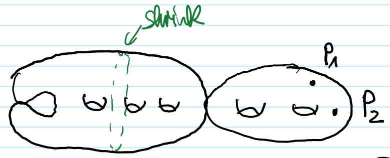
b) Given Γ_1, Γ_2, what is $\overline{M}^{\Gamma_1} \cap \overline{M}^{\Gamma_2}$? $\rightsquigarrow$ related to cup prod. $[\overline{M}^{\Gamma_1}] \cup [\overline{M}^{\Gamma_2}]$.

For a)

Exa

Γ

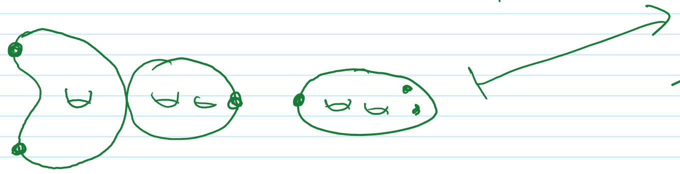


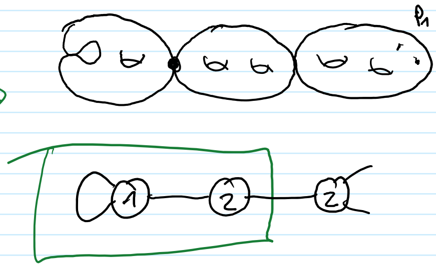


$\in M^\Gamma$

Q Can you write down / draw curve in $\overline{M}^\Gamma \setminus M^\Gamma$?

$\xi_\Gamma : \overline{M}_{3,3} \times \overline{M}_{2,3} \longrightarrow \overline{M}_{6,2}$





Γ'