Integrals of ψ-classes on spaces of K-differentials & twisted double-ramification cycles

j.w. in progress w/ M. Costantini, A. Sauvaget

Overview

1. Integrals of ψ-classes on $\overline{\mathcal{M}}_g(a)$ and $DR_g(a)$
2. Euler characteristics of minimal strata of differentials
3. Spin refinements

1. Integrals of ψ-classes on $\overline{\mathcal{M}}_g(a)$ and $DR_g(a)$

log convention!

Let $g,n,K \geq 0$ with $2g-2+n>0$. Let $a \in \mathbb{Z}^n$ with $\sum_{i=1}^n a_i = K(2g-2+n)$.

$$\rightsquigarrow \mathcal{M}_g(a) = \left\{ (C, p_1, \ldots, p_n) : (\omega_C^{\log})^{\otimes K} \cong \mathcal{O}_C\left(\sum_{i=1}^n a_i p_i\right) \right\} \subseteq \mathcal{M}_{g,n}$$

moduli of smooth curves

$$\Leftrightarrow \omega_C^{\otimes K} \cong \mathcal{O}_C\left(\sum_{i=1}^n (a_i - K) p_i\right)$$

$\Leftrightarrow$ $\exists$ meromorphic K-differential η on C with $\operatorname{div}(\eta) = \sum_i (a_i - K) p_i$

$\rightsquigarrow \overline{\mathcal{M}}_g(a) \subseteq \overline{\mathcal{M}}_{g,n}$ closure — stratum of K-differentials

moduli of stable curves

Many people have studied these strata:

→ modular characterization of points in closure

- Bainbridge-Chen-Gendron-Grushevsky-Möller '18 (K=1), '19 (K>1)
- Benirschke '20 (K=0)

→ smooth modular compactification via multi-scale differentials

- BCGGM '19 (K=1)
- Costantini-Möller-Zachhuber '19 (K>1)

→ characterization of connected components

- Kontsevich-Zorich '03, Boissy '15 (K=1)
- Lanneau '08 (K=2)
- Chen-Gendron '21 (K>1)

→ ... many more contributions: intersection theory, theory of flat surfaces, Teichmüller theory, ...

In the first part of the talk, we want to study a very concrete question:

Q Can we give a formula for

$$B_g(a) = \int_{[\overline{\mathcal{M}}_g(a)]} \psi_1^{2g-3+n} \quad ?$$

$\psi_1 = c_1(\mathbb{L}_1)$, $\mathbb{L}_1$ line bd. on $\overline{\mathcal{M}}_{g,n}$, $\mathbb{L}_1|_{(C,p_1,\ldots,p_n)} = T^*_{p_1}C$.

Motivation Q&A

- Why power $2g-3+n$ of ψ_1?
  ↳ want to get nonzero answer:

Thm [P'06, FP'18, S'18, M, V'86, BCGGM'19, ...]
The space $\overline{\mathcal{M}}_g(a)$ has dimension $2g-3+n$, except for $a \in (K\cdot\mathbb{Z}_{>0})^n$, in which case

$\overline{\mathcal{M}}_g(\frac{a}{K}) \subseteq \overline{\mathcal{M}}_g(a)$
stratum of holomorphic 1-differentials

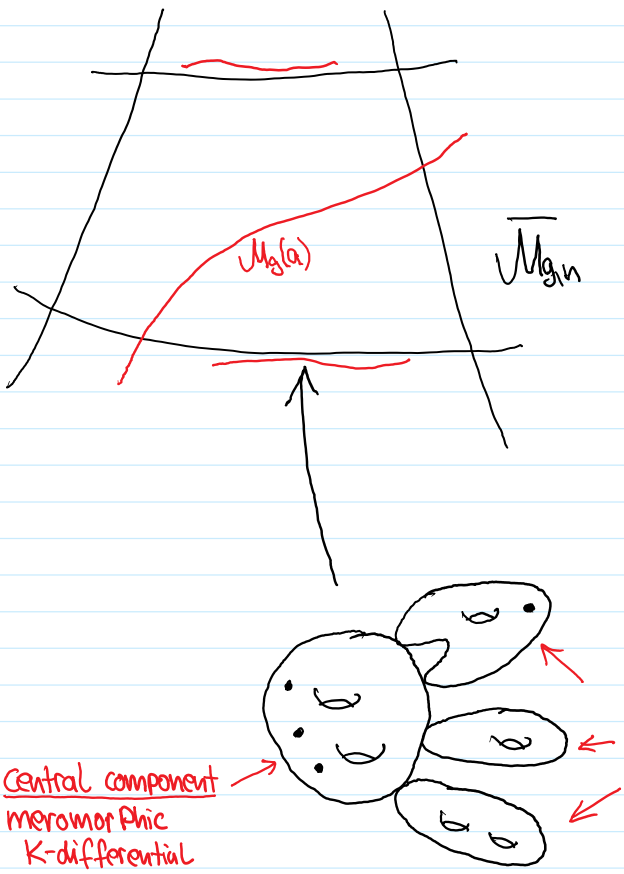
has dimension $2g-2+n$. [except: $a=\underline{0} \rightsquigarrow \overline{\mathcal{M}}_g(a) = \overline{\mathcal{M}}_{g,n}$]

- Why study B_g at all?
  ↳ natural question in study of fund. class $[\overline{\mathcal{M}}_g(a)]$
  ↳ numbers appear in recursive formulas for
    - ▷ Volumes of moduli spaces of flat surfaces [Sauvaget '20]
    - ▷ Euler characteristics of minimal strata $\overline{\mathcal{M}}_g(2g-1)$ of holomorphic abelian differentials $^{2g-2}$ [see part 2 below]
  ↳ good formula $\rightsquigarrow$ asymptotic analysis

- What about more general monomials in ψ-classes?
  ↳ harder question, future work

Main tool Relationship between $\overline{\mathcal{M}}_g(a)$ and the

Main tool: Relationship between $\overline{\mathcal{M}}_g(a)$ and the double-ramification cycle $DR_g(a)$

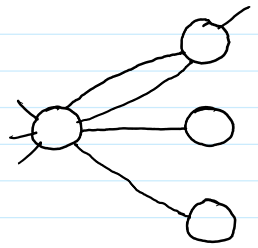


Farkas-Pandharipande '18 defined alternative compactif. of $\mathcal{M}_g(a)$:

$\widetilde{\mathcal{H}}_g^K(a)$ moduli space of twisted K-differentials

central component
meromorphic K-differential

outlying components
K-th power of holomorphic abelian differential



simple star graph

Thm (Conjecture A) [Janda-Pandharipande-Pixton-Zvonkine '18, S.'18, Holmes-S.'19, Bae-Holmes-Pandharipande-S.-Schwarz '20]

For $K>0$, $a \in \mathbb{Z}^n \setminus (K\mathbb{Z}_{>0})^n$ w/ $\sum_i a_i = K(2g-2+n)$:

$$\sum_{\substack{Z \subseteq \widetilde{\mathcal{H}}_g^K(a) \\ \text{component}}} m_Z \cdot [Z] = DR_g(a) \in H^{2g}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

explicit pos. integer weights, $m_Z = 1$ for $Z \subseteq \overline{\mathcal{M}}_g(a)$

twisted double-ramification cycle

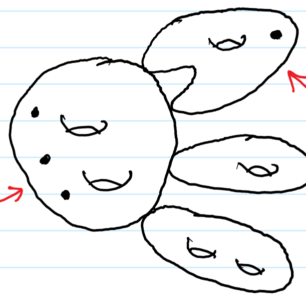
Here the cycle $DR_g(a)$:

- has an explicit formula in the tautological ring $RH^*(\overline{\mathcal{M}}_{g,n}) \subseteq H^*(\overline{\mathcal{M}}_{g,n})$ [Pixton]
- is a cycle-valued polynomial of degree at most $2g$ in the entries of the vector a [Pixton-Zagier]

Lem For $k\geq 1$ with $a_1<0$ or $k\nmid a_1$ or $a_1 > k(2g-1)$ $(\star)$ we have $\mathcal{B}_g(a) = \mathcal{A}_g(a)$ for

$$\mathcal{A}_g(a) = \int_{DR_g(a)} \psi_1^{2g-3+n}.$$

Proof Condition $(\star)$ guarantees that in component Z of $\widetilde{\mathcal{H}}_g^k(a)$ with $Z \not\subseteq \overline{\mathcal{M}}_g(a)$, marking p_1 is on the central component.



↑ have pos. dim'l moduli $\Rightarrow \psi_1^{2g-3+n}|_Z = 0$

Conj. A: $\int_{\overline{\mathcal{M}}_g(a)} \psi_1^{2g-3+n} = \int_{DR_g(a)} \psi_1^{2g-3+n}$. $\square$

What is known about $\mathcal{A}_g$?

Thm [Buryak-Shadrin-Spitz-Zvonkine '15]
For $a\in\mathbb{Z}^n$ with $\sum_i a_i = 0$ $(\Leftrightarrow k=0)$ we have

$$\mathcal{A}_g(a) = [z^{2g}] \frac{\prod_{i=2}^n \mathcal{S}(a_i z)}{\mathcal{S}(z)}, \quad \text{where } \mathcal{S}(z) = \frac{\sinh(z/2)}{z/2}.$$

↑ taking coeff. of z^{2g} in power series

What we can show:

Thm [Costantini-Sauvaget-S.]
For all g,n and $a\in\mathbb{Z}^n$, setting $k=\sum_i a_i/(2g-2+n)$, we have

$$\mathcal{A}_g(a) = [z^{2g}] \exp\left(\frac{a_1 z\cdot \mathcal{S}'(kz)}{\mathcal{S}(kz)}\right)\cdot \frac{\prod_{i=2}^n \mathcal{S}(a_i z)}{\mathcal{S}(z)\cdot \mathcal{S}(kz)^{2g-1+n}}.$$

Examples Computer demonstration

Proof sketch Characterize $\mathcal{A}_g$ by univ. properties, show them for formula

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Lem The functions $\mathcal{A}_g$ satisfy:

(a) $\mathcal{A}_g$ polynomial in $a_1, \ldots, a_n$, of degree at most $2g$, symmetric in $a_2, \ldots, a_n$. [Pixton-Zagier]

(b) $\mathcal{A}_g(a_1, \ldots, a_n, K) = \mathcal{A}_g(a_1, \ldots, a_n)$ $\quad \left[K = \frac{a_1 + \cdots + a_n}{2g-2+n}\right]$ $\quad$ $\pi^* DR_g(a_1, \ldots, a_n) = DR(a_1, \ldots, a_n, K)$

(c) $$a_1 \cdot \mathcal{A}_g(a_1, \ldots, a_n, 0) = -\sum_{i=2}^{n} (a_i - K)\, \mathcal{A}_g(a_1, \ldots, a_i - K, \ldots, a_n) + \frac{1}{2} \sum_{j=0}^{K} j(K-j)\, \mathcal{A}_{g-1}(a_1, \ldots, a_n, -j, j-K)$$ $\left[K = \frac{a_1 + \cdots + a_n}{2g-1+n}\right]$

Splitting formula for ψ-classes on DR-cycles (☆☆)

(d) $$\mathcal{A}_g \Big|_{\sum a_i = 0} = [z^{2g}] \frac{\prod_{i=1}^{n} \mathcal{S}(a_i z)}{\mathcal{S}(z)}.$$ [BSSZ'15]

Lem
- Properties (a)–(d) above together with initial data $\mathcal{A}_0, \mathcal{A}_1$ characterize functions $\mathcal{A}_g$ completely.

  combinatorial argument, $\mathcal{A}_g = \sum c_{d,\mu} a_1^d e_\mu(a)$, determine $c_{d,\mu}$ inductively
- Formula from Thm above satisfies all these properties. Analysis II-exercise

$\square$

Some more work: get formulas for $\int_{[\overline{\mathcal{M}}_g(a)]} \psi_1^u \quad \forall a \text{ w/ } K>0 \quad \forall u$.

2. Euler characteristics of minimal strata of differentials

For $K=1$, [Costantini-Möller-Zachhuber '20] computed the orbifold Euler characteristic $\chi(\mathcal{M}_g(a))$ for many examples of g, a.

Idea $\overline{\mathcal{M}}^m$ compact, smooth orbifold $\Rightarrow \chi(\overline{\mathcal{M}}) = (-1)^m \int_{\overline{\mathcal{M}}} c_{top}(\Omega^1_{\overline{\mathcal{M}}})$

$\mathcal{M} \subseteq \overline{\mathcal{M}}$ open such that $D = \overline{\mathcal{M}} \setminus \mathcal{M}$ n.c. divisor $\Big\} \Rightarrow \chi(\mathcal{M}) = (-1)^m \int_{\overline{\mathcal{M}}} c_{top}(\Omega^1_{\overline{\mathcal{M}}}(\log D))$

Logarithmic cotangent bundle

Apply this to BCGGM-compactification

$$\overline{\mathcal{M}} = \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(a) \supseteq \mathcal{M}_g(a) = \mathcal{M}$$

$\mu = (a_1 - 1, \ldots, a_n - 1)$

μ	(0)	(2)	$(1,1)$	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$

$\mu = (a_1 - 1, \dots, a_n - 1)$

[CMZ'20]

μ	(0)	(2)	$(1,1)$	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$
$\chi(B)$	$-\frac{1}{12}$	$-\frac{1}{40}$	$\frac{1}{30}$	$-\frac{55}{504}$	$\frac{16}{63}$	$\frac{15}{56}$	$-\frac{6}{7}$	$\frac{11}{3}$
μ	(6)	$(5,1)$	$(4,2)$	$(3,3)$	$(4,1,1)$	$(3,2,1)$	$(2,2,2)$	(8)
$\chi(B)$	$-\frac{1169}{720}$	$\frac{27}{5}$	$\frac{76}{15}$	$\frac{188}{45}$	$-\frac{200}{9}$	$-\frac{96}{5}$	$-\frac{187}{10}$	$-\frac{4671}{88}$

TABLE 1. Euler characteristics of some holomorphic strata

For $n=1$, $a=(2g-1)$, we can give explicit formula, no longer involving intersection numbers.

"Theorem" [CSS]

For any $g \geq 1$ we have that $\chi(\mathcal{M}_g(2g-1))$ is given by

$$(-1)^g \cdot 2 \cdot g \cdot a_g - \sum_{g^T=1}^{g-1} 2g^T \cdot (-1)^{g^T-1} \sum_{\substack{\underline{g}=(g_1,\dots,g_m) \\ \sum g_i = g^T}} \frac{(-1)^m}{m!} (2g-1)^{2g-2g^T+m-2} \cdot \mathcal{A}_{g-g^T}(2g-1, -2g_1+1, \dots, -2g_m+1) \cdot \prod_{i=1}^{m} (2g_i-1)! \cdot b_{g_i}$$

where the coefficients a_g and b_{g_i} are determined by:

- $\exists\, \mathcal{F} \in \mathbb{Q}[[t^2]]$ with $[t^{2g}]\, \overline{S}(t)^{-1} = \frac{1}{(2g)!} [t^{2g}]\, \mathcal{F}(t)^{2g}$ s.t.
$$\mathcal{F}(t) = 1 + \sum_{g>0} (2g-1)\, a_g\, t^{2g}, \quad \text{for } \overline{S}(t) = \frac{\sin(z/2)}{z/2}$$
- $$\overline{S}(t)^{-1} = 1 + \sum_{g>0} b_g\, t^{2g},$$

Idea of Proof

- [CMZ'20] give formula for $\chi(\mathcal{M}_g(a))$: graph sum w/ inters. numbers
- minimal stratum $\Rightarrow$ shape of graphs somewhat restricted
- can use divisorial relations on spaces $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(a)$ to manipulate graph sum
- result = sum in terms of
  - integrals of ξ^{top} on minimal strata: a_g, b_{g_i} [Sauvaget'18]
  - integrals of ψ_1^{top} on merom. strata: $\mathcal{A}_g$

$\square$

3. Spin refinements

For k odd, $a_1,\ldots,a_n$ odd and $(C,p_1,\ldots,p_n)\in\mathcal{M}_g(a)$:

$$\omega_C^{\otimes k}\cong\mathcal{O}_C\Big(\sum(a_i-k)p_i\Big)\iff\omega_C\cong\Big(\underbrace{\omega_C^{-(k-1)/2}\Big(\sum\frac{a_i-k}{2}p_i\Big)}_{=:\,\mathcal{L}\ \text{spin structure}}\Big)^{\otimes 2}$$

$\Rightarrow$ $p_{\mathcal{L}}=(h^0(\mathcal{L})\ \mathrm{mod}\ 2)$ is a deformation invariant

spin parity

$\Rightarrow$ $\mathcal{M}_g(a)=\mathcal{M}_g(a)^{\mathrm{even}}\sqcup\mathcal{M}_g(a)^{\mathrm{odd}}\subseteq\mathcal{M}_{g,n}$

Def $[\overline{\mathcal{M}}_g(a)]^{\mathrm{spin}}=[\overline{\mathcal{M}}_g(a)^{\mathrm{even}}]-[\overline{\mathcal{M}}_g(a)^{\mathrm{odd}}]\in H^*(\overline{\mathcal{M}}_{g,n})$

⚠ **Warning** This is different from notion of spin for strata of k-differentials in [Chen–Gendron '21].

Q What is $\mathcal{B}_g^{\mathrm{spin}}(a)=\int_{[\overline{\mathcal{M}}_g(a)]^{\mathrm{spin}}}\psi_1^{2g-3+n}$?

Conjecture 1 There exists a cycle $\mathrm{DR}_g^{\mathrm{spin}}(a)\in RH^{2g}(\overline{\mathcal{M}}_{g,n})$, given by a cycle-valued polynomial of degree $2g$ in a, equivariant w.r.t. permutations of markings such that for $a_1<0$ or $k\nmid a_1$ or $a_1>k(2g-1)$ $(\star)$ we have $\mathcal{B}_g^{\mathrm{spin}}(a)=\mathcal{A}_g^{\mathrm{spin}}(a)$ for

$$\mathcal{A}_g^{\mathrm{spin}}(a)=\int_{\mathrm{DR}_g^{\mathrm{spin}}(a)}\psi_1^{2g-3+n}.$$

Conjecture 2 We have

$$\mathcal{A}_g^{\mathrm{spin}}(a)=2^{-g}\,[z^{2g}]\,\exp\left(\frac{a_1 z\,\mathcal{S}'(kz)}{\mathcal{S}(kz)}\right)\frac{\cosh(z/2)}{\mathcal{S}(z)}\,\frac{\prod_{i=2}^{n}\mathcal{S}(a_iz)}{\mathcal{S}(kz)^{2g-1+n}}.$$

Rmk 1 We have a sketch of proof that Conj. 1 $\Rightarrow$ Conj. 2, assuming natural splitting properties for $[\overline{\mathcal{M}}_g(a)]^{\mathrm{spin}}$.

Rmk 2 We have an explicit proposal of a formula for $\mathrm{DR}_g^{\mathrm{spin}}$ [Appendix]

Conjecture 2 for $\int_{\mathrm{DR}^{\mathrm{spin}}}\psi^{\mathrm{top}}$

[Holmes–Orecchia–Pandharip.–S.–Tseng] variant of Conjecture A for moduli $\mathcal{S}_{g,n}$ of spin structures, compactifying

variant of conjecture A for moduli $\overline{S}_{g,n}$ of spin structures, compactifying
$$\left\{(C,\underline{p},\mathcal{L}) \mid \mathcal{L} = \omega_C^{-(k-1)/2}\left(\sum \frac{a_i - k}{2} p_i\right)\right\}$$

Computer check

$$DR_g^{spin}(a)$$

[Giacchetto-Kramer-Lewański '21]
Spin version of r-spin Hurwitz numbers, ELSV-formula involving spin Chiodo class,
polynomial in $r \gg 0$
set $r = 0$
DR_g^{spin}

[Pixton-Zagier]
Combinatorially natural variant of formula for DR_g, involving half-integer weightings, polynomiality in a

Thank you for your attention!

"Theorem" [CSS]

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$$(-1)^g \cdot 2g \cdot a_g - \sum_{g^T=1}^{g-1} 2g^T \cdot (-1)^{g^T-1} \sum_{\substack{\underline{g}=(g_1,\dots,g_m) \\ \sum g_i = g^T}} \frac{(-1)^m}{m!} (2g-1)^{2g-2g^T+m-2} \cdot \mathcal{A}_{g-g^T}(2g-1, -2g_1+1, \dots, -2g_m+1) \cdot \prod_{i=1}^{m} (2g_i-1)! \cdot b_{g_i}$$

where the coefficients a_g and b_{g_i} are determined by:

- $\exists\, \mathcal{F} \in \mathbb{Q}[[t^2]]$ with $[t^{2g}]\, \bar{S}(t)^{-1} = \frac{1}{(2g)!} [t^{2g}]\, \mathcal{F}(t)^{2g}$ s.t.
$$\mathcal{F}(t) = 1 + \sum_{g>0} (2g-1)\, a_g\, t^{2g}, \quad \text{for } \bar{S}(t) = \frac{\sin(z/2)}{z/2}$$
- $\bar{S}(t)^{-1} = 1 + \sum_{g>0} b_g\, t^{2g}$,

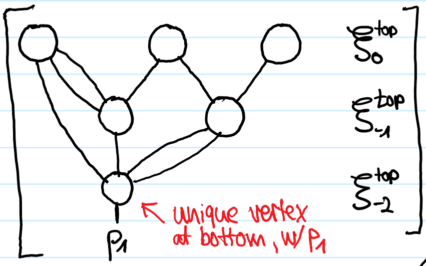
Proof sketch

Spaces $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(a)$ carry linebdle. $\mathcal{O}(-1)$ w/ $\mathcal{O}(-1)|_{(C,p_1,\dots,p_n)} = \mathbb{C}\cdot\eta$

C smooth $\operatorname{div}(\eta) = \sum (a_i - 1) p_i$

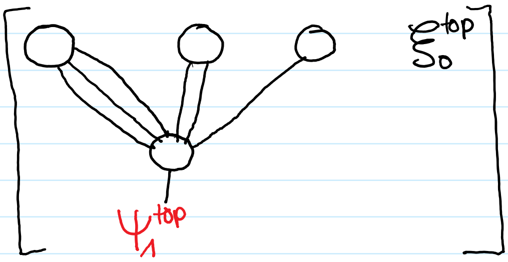
$\rightsquigarrow \xi = c_1(\mathcal{O}(-1))$

$$\chi(\mathcal{M}_g(2g-1)) \overset{[CMZ'20]}{=} 2g \cdot \underbrace{\int_{\mathbb{P}\Xi\overline{\mathcal{M}}_{g,1}(2g-1)} \xi^{top}}_{\overset{[S'18]}{=} (-1)^g \cdot a_g} + \sum_{\Gamma} cst_\Gamma \cdot [\ \cdots\ ]$$

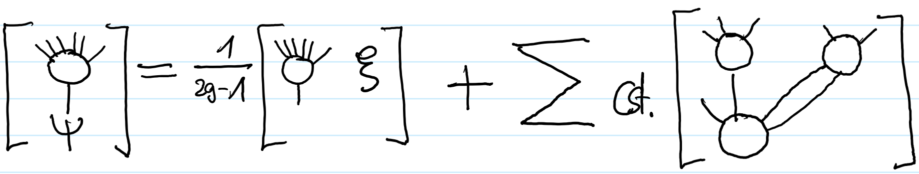


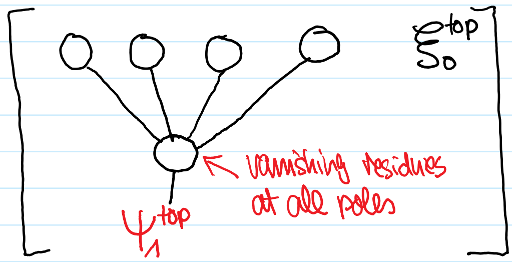
$(\star)$

$$(\star) = \sum_{\Gamma'} cst_{\Gamma'} \cdot [\ \cdots\ ]$$



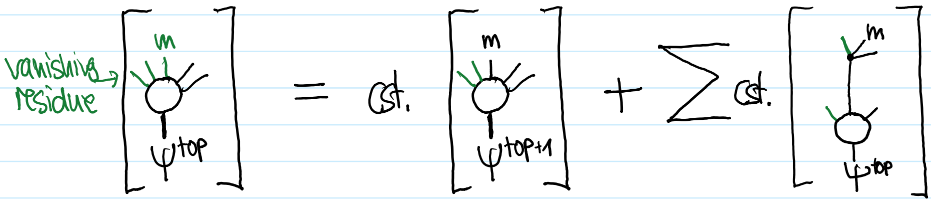
using: $[\ \cdots\ ] = \frac{1}{2g-1} [\ \cdots\ \xi\ ] + \sum cst. [\ \cdots\ ]$

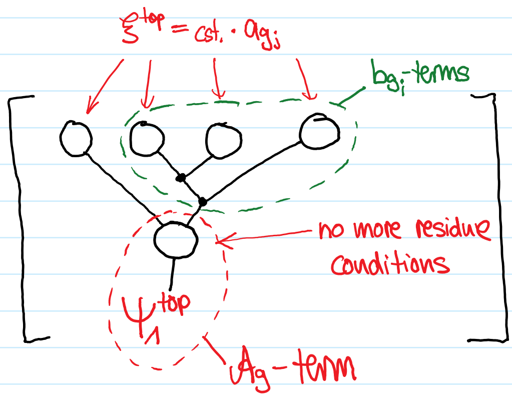


$(\star) = \sum_{\Gamma'} cst_{\Gamma'} \cdot$ 

$\leftarrow$ ξ_0^{top} vanishes on non-minimal strata, equals $cst. \cdot a_{g_j}$ on minimal strata

Use relation to start eliminating residue conditions:



$(\star) = \sum_{\Gamma''} cst_{\Gamma''} \cdot$ 

$\square$

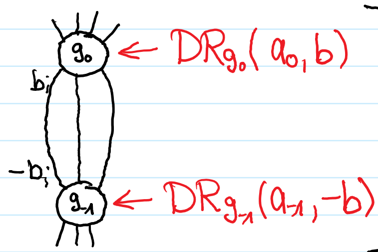
In progress: analyze asymptotics of $\chi(\mathcal{M}_g(2g-1))$.

<u>Lem</u> (Splitting formula (☆☆))

Let $g,n \geq 0$ w/ $2g-2+n>0$, $a \in \mathbb{Z}^n$ w/ $\sum_i a_i = K(2g-2+n)$, $s,t \in \{1,\ldots,n\}$ distinct, then:

$$(a_s\psi_s - a_t\psi_t)\,\mathrm{DR}_g(a)$$

$$= \quad \psi_{s,t}(\Gamma)\cdot \frac{\prod_{j=1}^{m} b_j}{m!} \cdot \Bigg[\quad \Bigg]$$



$b_\Gamma = K(2g_0-2+n_{v_0}) - |a_0|$

$$\psi_{s,t}(\Gamma) = \begin{cases} 1, & s \text{ at } v_{-1},\ t \text{ at } v_0 \\ -1, & t \text{ at } v_{-1},\ s \text{ at } v_0 \\ 0, & \text{otherwise} \end{cases}$$

<u>Proof</u> combines splitting formula for $\overline{\mathcal{M}}_g(a)$ [Sauvaget'20], Conjecture A and polynomiality. $\square$

The following is a proposal for DR_g^{spin}, based on work of [Giachetto-Kramer-Lewański '21]:

$$DR_g^{r,spin,\bullet}(a) = 2^{-g} r^{2g} 2^{-g} \cdot \sum_{\Gamma \in \mathcal{G}_{g,n}} \sum_{w \in W^{odd}_{\Gamma,r,K}} \frac{(r/2)^{-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \xi_{\Gamma *}\Bigg[\prod_{v \in V(\Gamma)} e^{-\sum_{m\geq 1} (-1)^{m-1} \frac{B_{m+1}(K/r)}{m(m+1)} K_m(v)}$$
$$\cdot \prod_{i=1}^{n} e^{\sum_{m\geq 1} (-1)^{m-1} \frac{B_{m+1}(a_i/r)}{m(m+1)} \psi_{h_i}^m}$$
$$\cdot \prod_{\substack{e \in E(\Gamma) \\ e=(h,h')}} \frac{1 - e^{\sum_{m\geq 1}(-1)^{m-1} \frac{B_{m+1}(w(h)/r)}{m\cdot(m+1)} [\psi_h^m - (-\psi_{h'})^m]}}{\psi_h + \psi_{h'}} \Bigg]$$

↳ polynomial in r even for $r \gg 0$

Complex codimension g part

$$DR_g^{spin}(a) = DR_g^{r,spin,g}(a)\Big|_{r=0}$$