Enumerative invariants from log intersection numbers

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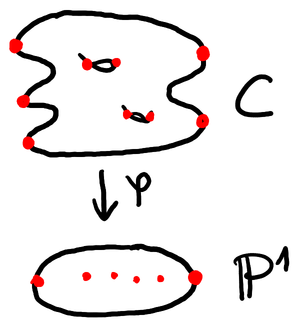
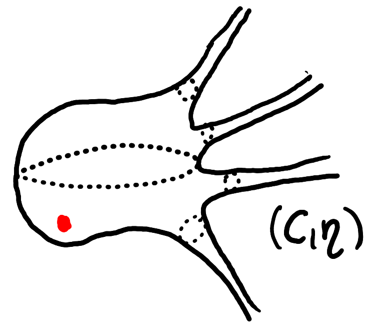
§0 Motivation

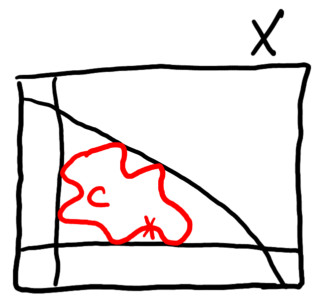
Enumerative Geometry count geometric objects w/ specified properties



Hurwitz numbers

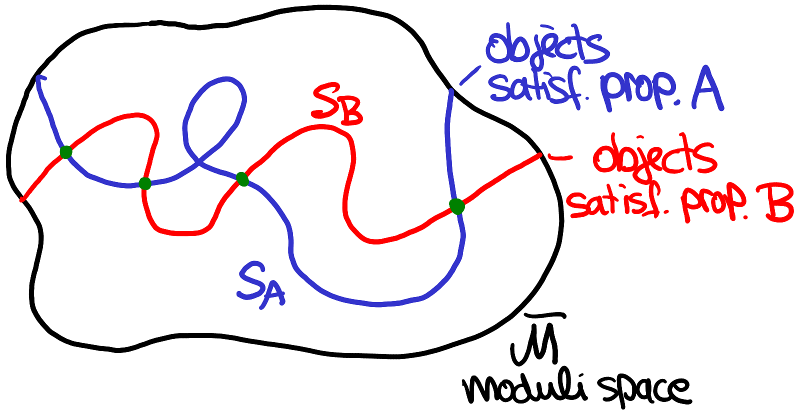


Merom. differentials w/ zero, pole, period conditions



Maps to target X w/ incidence & tangency conditions

Intersection theory (Pipe dream) enumerative count as intersection number



$$\boxed{N_{AB} = \int_{\overline{M}} S_A \cdot S_B}$$

Problems
- Extra intersections in boundary of $\overline{M}$
- Compactifying & computing S_A, S_B

} wrong $\overline{M}$!

Goal Show how log intersection theory can help

§1. Compactifying strata of K-differentials

Fix $g, n, K \geq 0$ and $A = (a_1, \dots, a_n) \in \mathbb{Z}^n$ w/ $\sum a_i = K \cdot (2g-2+n)$

$$DR_g^\circ(A) = \left\{ (C, p_1, \dots, p_n) \in \mathcal{M}_{g,n} \;\middle|\; \begin{array}{l} \exists \text{ merom. } K\text{-diff. } \eta \text{ on } C \\ \text{with } \operatorname{div}(\eta) = \sum_{i=1}^n (a_i - K) p_i \end{array} \right\}$$

log convent.

$$= \left\{ (C, p_1, \dots, p_n) \in \mathcal{M}_{g,n} \;\middle|\; (\omega_C^{\log})^{\otimes K} \cong \mathcal{O}_C\left(\sum a_i p_i\right) \right\}$$

strata of K-differentials

$K=0$: merom. functions $\rightsquigarrow$ Hurwitz numbers

$K=1$: translation surfaces $\Big\}\rightsquigarrow$ geodesics,

$K=2$: half-translation surfaces $\Big\}$ Masur-Veech volumes

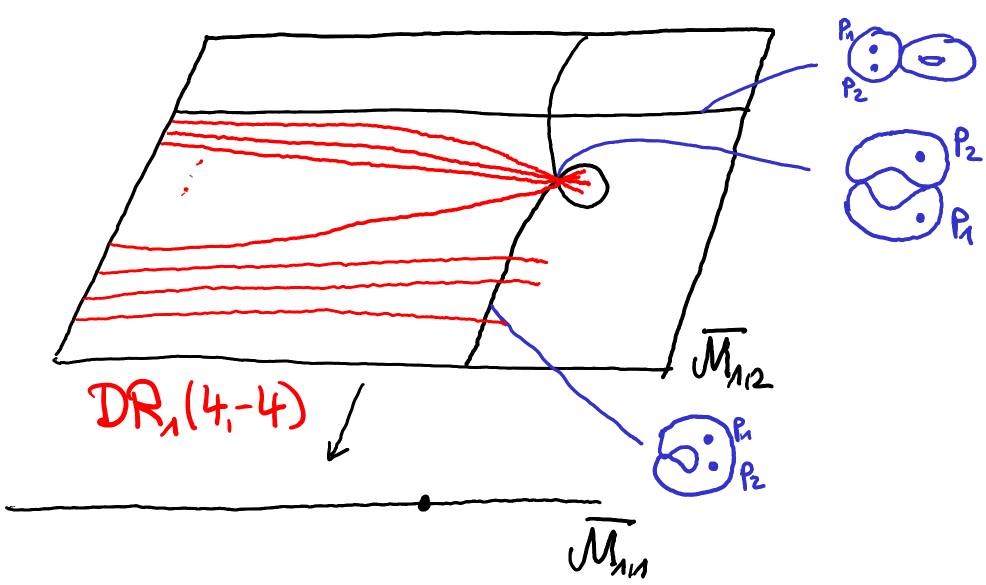
$K \to \infty$: flat surfaces w/ real cone angles [Sauvaget]

Q How to compactify $DR_g^\circ(A)$?

moduli space of stable curves

Idea 1 Take closure $\overline{DR_g^\circ(A)}$ inside $\overline{\mathcal{M}}_{g,n}$.

Problem result is highly singular!



Idea 2 Construct smooth compactification of $DR_g^\circ(A)$ by hand

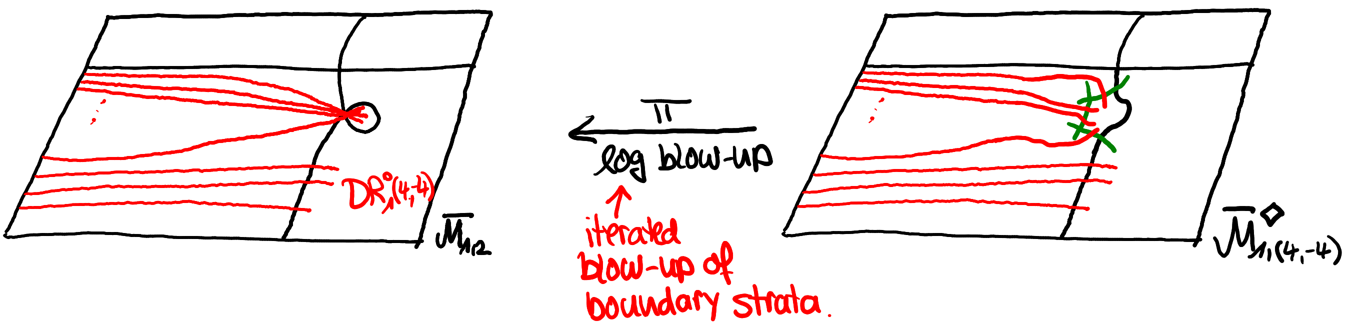
Thm [Bainbridge–Chen–Gendron–Grushevsky–Möller ($k=1$)
Costantini–Möller–Zachhuber ($k>1$)] $\hat{A}=(a_1-k,\dots,a_n-k)$

$\exists$ smooth proper DM stack $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\hat{A})$ compactifying $DR_g^\circ(A)$ and parameterizing a stable curve C together with a multi-scale k-differential

enhanced level graph, twisted differentials, prong matchings, rescaling ensembles, ...

Questions $k=0$? ambient space $\overline{\mathcal{M}}$? simpler definition?

Idea 3 Take closure of $DR_g^\circ(A)$ inside blow-up of $\overline{\mathcal{M}}_{g,n}$



Thm [Chen–Grushevsky–Holmes–Möller–S.]

(a) $\exists$ natural log blow-up $\pi: \overline{\mathcal{M}}^\diamond_{g,A} \longrightarrow \overline{\mathcal{M}}_{g,n}$ such that

$$\overline{DR_g^\circ(A)}^{\overline{\mathcal{M}}^\diamond_{g,A}} \cong \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\hat{A}) \qquad \text{for } k=1.$$

strict transform under π

(b) modular interpretation as log stack

$$\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\hat{A}) = \left\{ \left(C, p_1,\dots,p_n, (\omega^{\log})^{\otimes k} \xrightarrow[\sim]{\varphi} \mathcal{O}_C\left(\textstyle\sum a_i p_i\right)(\alpha)\right) \right\} \quad (\star)$$

Piecew. linear function on C

+ Global Residue Condition (GRC)

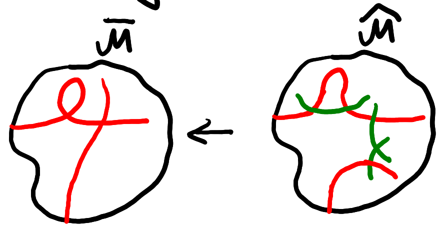
moduli functor over LogSch

For arbitrary $K \geq 0$:

$\underbrace{\widehat{DR}_g(A)}_{\substack{:= \text{RHS of } (\star) \\ \text{w/o GRC}}} \subseteq \overline{\mathcal{M}}^{\diamond}_{g,A}$ closed substack, virtual fundamental class

<u>Def</u> (logarithmic Chow ring)
$\overline{\mathcal{M}}$ smooth space, $D \subseteq \overline{\mathcal{M}}$ normal-crossing divisor

$$\log CH^*(\overline{\mathcal{M}}, D) := \varinjlim_{\substack{\widehat{\mathcal{M}} \to \overline{\mathcal{M}} \\ \text{log blow-up}}} CH^*(\widehat{\mathcal{M}})$$



$$= \left\{ (\widehat{\mathcal{M}} \xrightarrow{\pi} \overline{\mathcal{M}}, \alpha \in CH^*(\widehat{\mathcal{M}}) \right\} \Big/ \begin{array}{l} (\widehat{\mathcal{M}}, \alpha) \sim (\widehat{\widehat{\mathcal{M}}}, \beta) \\ \text{if } \widehat{\widehat{\mathcal{M}}} \xrightarrow{\widehat{\pi}} \widehat{\mathcal{M}} \xrightarrow{\pi} \overline{\mathcal{M}} \\ \text{and } \beta = \widehat{\pi}^* \alpha. \end{array}$$

<u>Intuition</u>
$\log CH^*(\overline{\mathcal{M}})$ refines $CH^*(\overline{\mathcal{M}})$, describes intersect. theory on all log blow-ups $\widehat{\mathcal{M}} \to \overline{\mathcal{M}}$ simultaneously.

<u>Def</u> (logarithmic double ramification cycle)

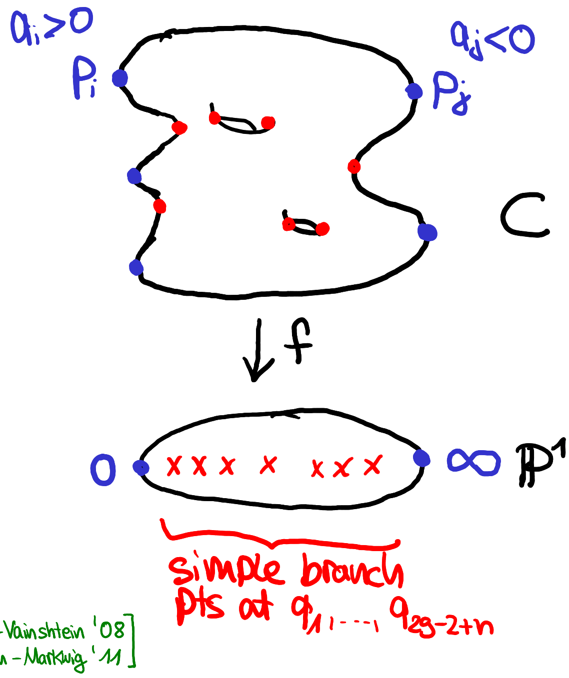
$$\log DR_g(A) := \left[\overline{\mathcal{M}}^{\diamond}_{g,A}, [\widehat{DR}_g(A)]^{vir} \right] \in \log CH^*(\overline{\mathcal{M}}_{g,n}).$$

Upshot: nice cycle, compactifying stratum of K-different. η with $\mathrm{div}(\eta) = \sum_{i=1}^{n} (a_i - K) p_i$

<u>Rest of talk</u>: Squeeze enumerative invariants from $\log DR_g(A)$.

§2 Double Hurwitz numbers

[Cavalieri–Markwig–Ranganathan, $k=0$]



Given $A=(a_1,\dots,a_n)\in\mathbb{Z}^n$ w/ $\sum a_i=0$:

$$H_g(A)=\sum_f \frac{1}{|\mathrm{Aut}(f)|}$$

Properties

- Piecewise polynomial in A [Shadrin–Shapiro–Vainshtein '08, Cavalieri–Johnson–Markwig '11]
  $\rightsquigarrow$ chamber decomposition and wall-crossing formulas
- Calculation via character formulas, cut-and-join equations, topological recursion, tropical geometry [Borot–Do–Karev–Lewański–Moskovsky '20, Cavalieri–Johnson–Markwig '10]
- GJV conjecture: ELSV-type formula
  $\rightsquigarrow$ proposals via Chiodo class on $\overline{\mathcal{M}}_{g,n}$ [Do–Lewański '20, BDKLM '20]

<u>Q</u> Can we get $H_g(A)$ from naive intersect. theory?

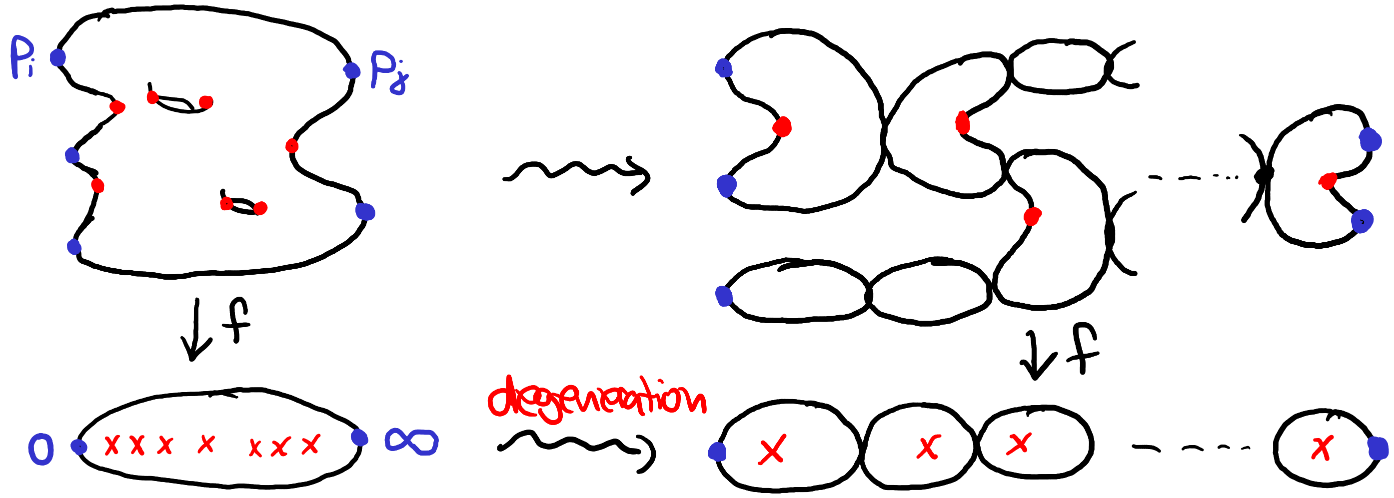
<u>Thm</u> [Cavalieri–Markwig–Ranganathan '22]

$$H_g(A)=\int_{\overline{\mathcal{M}}^{\diamond}_{g,A}} \log DR_g(A)\cdot br_{g,A}$$

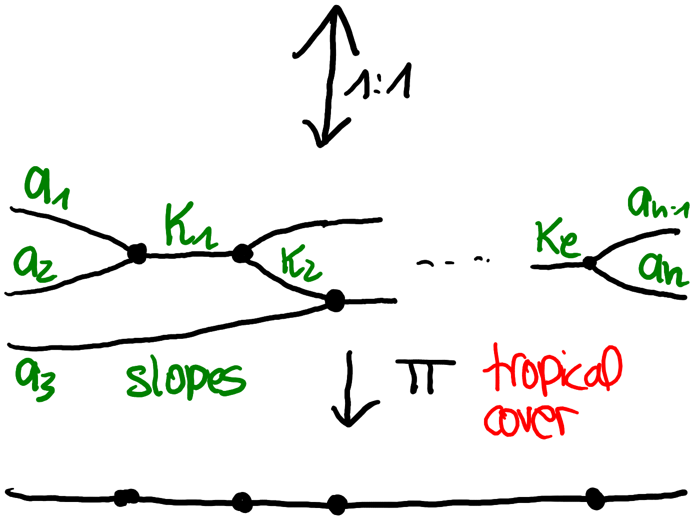
$\in \log CH^{2g-3+n}(\overline{\mathcal{M}}_{g,n})$ encoding fixed branch conditions

$br_{g,A}$ = lin. comb. of codim $2g-3+n$ bdry strata of $\overline{\mathcal{M}}^{\diamond}_{g,A}$

Idea of proof: degenerating target $(\mathbb{P}^1, 0, \infty)$



$$H_g(A)$$
$$\| \; [CJM'10]$$
$$\sum_{\pi \text{ trop. cover}} \frac{\prod_j K_j}{|\mathrm{Aut}(\pi)|}$$
$$\| \; \text{explicit analysis}$$
$$\int_{\overline{\mathcal{M}}^{\diamond}_{g,A}} \log DR_g(A) \cdot br_{g,A}$$



□

§3. Generalization to K-differentials

$$H_g^K(A) := \int_{\overline{\mathcal{M}}^{\diamond}_{g,A}} \log DR_g(A) \cdot br_{g,A} \qquad [CMR'22]$$

$$H_g^K(A,e) := \int_{\overline{\mathcal{M}}^{\diamond}_{g,A}} \log DR_g(A) \cdot br^{c}_{g,A} \cdot \psi_1^{e_1} \cdots \psi_n^{e_n} \qquad [CMS'24]$$

$c = 2g-3+n-|e|$

↳ K-leaky double Hurwitz numbers / descendants

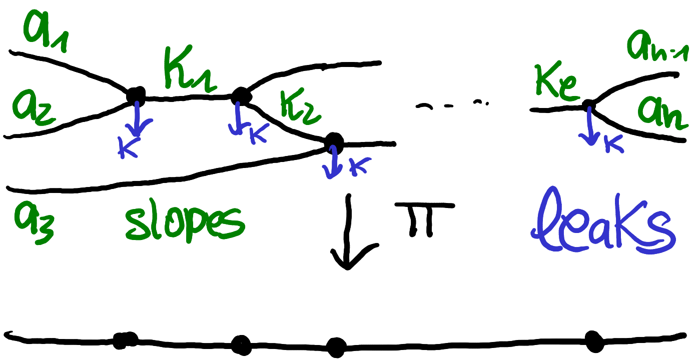
Idea $\quad H_g^K(A) \xleftrightarrow[\text{interpolates}]{H_g^K(A,e)} \int_{\overline{\mathcal{M}}_{g,n}} DR_g(A) \cdot \psi_1^{e_1} \cdots \psi_n^{e_n}$

[Buryak–Shadrin–Spitz–Zvonkine '15
Costantini–Sauvaget–S. '21
Sauvaget '22]

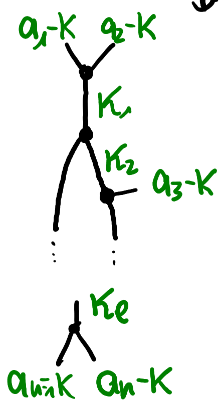
Thm [Cavalieri–Markwig–S. '24]

- $H_g^K(A,e)$ is piecewise polynomial of degree $4g-3+n-|e|$ in A
- Wall-crossing formula in genus 0
- Non-negativity in genus 0 $\rightsquigarrow$ in general: negative virtual boundary contributions.
- Tropical graph sum formula

$$H_g^K(A) = \sum_{\substack{\pi \\ K\text{-leaky cover}}} \frac{\prod_j K_j}{|\mathrm{Aut}(\pi)|}$$



rotate



enhanced level graphs [BCGGM] (ignoring GRC)

Big question Enumerative meaning of $H_g^k(A)$?

One idea: Counting k-differentials w/ period / residue cond.

Fix $A = (d, -b_1, \dots, -b_n) \in \mathbb{Z}^{n+1}$ w/ $|A| = n-1$, $d, b_1, \dots, b_n > 0$

$$DR_0^\circ(A) = \{(C, p_0, p_1, \dots, p_n) : \exists \eta \text{ 1-diff. on } C : \operatorname{div}(\eta) = (d-1)p_0 + \sum (-b_i - 1)p_i\}$$

$$\cup\!\!\!|$$

$$DR_0^\circ(A)^{\vec{r}} = \{ - - - : (\operatorname{Res}_{p_1}\eta, \dots, \operatorname{Res}_{p_n}\eta) \sim \vec{r}\}$$

$\vec{r} \in (\mathbb{C}^\times)^n$ vector of residues, $\sum r_i = 0$

Thm [Gendron-Tahar '20, Buryak-Rossi '23, Chen-Prado '23]

$\vec{r}$ general $\Rightarrow$ $|DR_0^\circ(A)^{\vec{r}}| = (d-1)(d-2)\cdots(d-(n-2))$

Cardinality

Thm [Cavalieri-Markwig-S. 'in prep.]

A as above $\Rightarrow$ $H_{g=0}^{k=1}(A) = (n-1)! \cdot (d-\frac{1}{2})(d-\frac{2}{2})\cdots(d-\frac{n-2}{2})$

structurally quite similar!

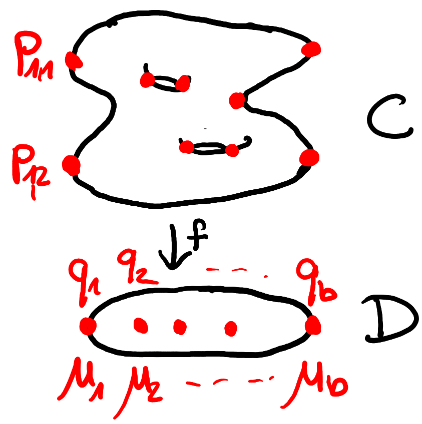
Questions

- Enumerative problem for arbitrary g, A ?
- Relationship to $H_g^k(A)$?

§4. Bonus: Arbitrary Hurwitz numbers in genus zero

admissible cover space [Harris-Mumf.]

$$\begin{array}{ccc} \overline{\mathrm{Hur}}_{g,\vec{\mu}} & \xrightarrow{\ \phi\ } & \overline{\mathcal{M}}_{g,r} \\ \delta \downarrow & (C \xrightarrow{f} D) \longmapsto (C, (p_{i\gamma})_{i,\gamma}) & \\ & \downarrow & \\ \overline{\mathcal{M}}_{0,b} & (D, (q_i)_i) & \end{array}$$



Def (Hurwitz number)

$$HN_{g,\vec{\mu}} := \deg(\delta)$$ $\rightsquigarrow$ can we get these via intersection theory?

Fact $\forall i,\gamma$ we have: $\delta^*\psi_i = \mu_{i,\gamma} \cdot \phi^*\psi_{i,\gamma}$

ψ at q_i multiplicity at $p_{i\gamma}$ ψ at $p_{i\gamma}$

Idea (Benirschke-Costantini-Mullane)

$$\Rightarrow \underbrace{\int_{\overline{\mathrm{Hur}}_{g,\vec{\mu}}} \phi^*(\psi_{i,\gamma}^{b-3})}_{=:(+)} = \frac{1}{\mu_{i\gamma}^{b-3}} \cdot \int_{\overline{\mathrm{Hur}}_{g,\vec{\mu}}} \delta^*\psi_i^{b-3}$$

$$= \frac{1}{\mu_{i\gamma}^{b-3}} \cdot \underbrace{\deg(\delta)}_{=HN_{g,\vec{\mu}}} \cdot \underbrace{\int_{\overline{\mathcal{M}}_{0,b}} \psi_i^{b-3}}_{=1}$$

$\Rightarrow$ Numbers (+) know about all Hurwitz numbers!

Problem How to get hands on $[\overline{\mathrm{Hur}}_{g,\vec{\mu}}]$?

Thm [Benirschke]

$$\begin{array}{ccc} \overline{\mathrm{Hur}}_{g,\vec{\mu}} & \xleftarrow{\ i\ }\!\!\!\!\longrightarrow & \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\tilde{A}) \\ (f: C \to \mathbb{P}^1) & \longmapsto & (C, df) \end{array}$$

merom. differential

as locus of exact differentials

$\rightsquigarrow$ formula for $i_*[\overline{\mathcal{H}ur}_{g,\vec{\mu}}]$ in terms of taut. classes.

<u>For genus $g=0$</u>

$$[\text{CGHMS'22}] \Longrightarrow \mathbb{P}\boxed{i}\overline{\mathcal{M}}_{0,n}(\hat{A}) \xrightarrow[\text{log blow-up}]{p} \overline{\mathcal{M}}_{0,n}$$

$$(+) = \int_{\overline{\mathcal{H}ur}_{g,\vec{\mu}}} \phi^*(\psi_{i_\delta}^{b-3}) = \int_{\mathbb{P}\boxed{i}\overline{\mathcal{M}}_{0,n}(\hat{A})} \underbrace{i_*[\overline{\mathcal{H}ur}_{g,\vec{\mu}}]\cdot \psi_{i_\delta}^{b-3}}_{\text{intersection in } \log CH^*(\overline{\mathcal{M}}_{0,n})}$$

<u>Hope</u> Recover structural properties / formulas of $HN_{g,\vec{\mu}}$ from log intersection theory.

Thanks for your attention!