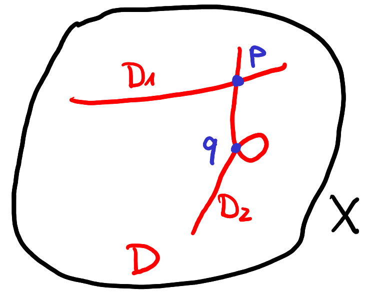
# Logarithmic tautological rings (j/w. R. Pandharipande, D. Ranganathan, P. Spelier)

## §0 Motivation

$(X,D)$ smooth space with normal crossings divisor (smooth log smooth)



E.g. $(X=\overline{\mathcal{M}}_{g,n}\ ,\ D=\partial\overline{\mathcal{M}}_{g,n})$

mod. space of stable curves

locus of singular curves

⇝ stratification $X=\bigsqcup_\sigma S_\sigma$ into loc. closed $S_\sigma\subseteq X$.

$\boxed{Q1}$ How to intersect classes $[\overline{S_\sigma}]\in CH^*(X)$ ?

⇝ nice combinatorial formalism?

E.g. $[D_1]\cdot[D_2]=[p]\in CH^2(X)$ above.

Next problem: sometimes $U=X\setminus D$ is canonical, but it's (partial) compactification $X$ is not!

Exa $U=\mathcal{A}_g$ : moduli space of princ. polarized abel. var of dim $g$

⇝ $X=\overline{\mathcal{A}}_g$ : different birat'l models ($\overline{\mathcal{A}}_g^{PC}$, $\overline{\mathcal{A}}_g^{2nd\ Vor}$, ...)

In good cases:

$$\begin{tikzcd}
& \exists\, \hat{X} \arrow[dl, dashed] \arrow[dr, dashed] & \\
X_1 & & X_2 \\
& U \arrow[ul, phantom, "\supseteq"] \arrow[ur, phantom, "\subseteq"] &
\end{tikzcd}$$

any two $X_1,X_2$ receive map from common space $\hat{X}$ and $\hat{X}\to X_i$ is a log blow-up

↑ think: iterated blow-up of smooth strata closures

Def (Holmes-Pixton-S.)

$(X,D)$ smooth nc pair.

$$⇝\ \log CH^*(X,D) := \lim_{\substack{(\hat{X},\hat{D})\to(X,D)\\ \text{log blow-up}}} CH^*(\hat{X})$$

↗ trans. maps = pullb. $\hat{\pi}^*$ $\quad \hat{\hat{X}}\xrightarrow{\hat{\pi}}\hat{X}\longrightarrow X$

logarithmic Chow ring

## Basic properties

- $\log CH^*(X,D) = \{(\hat{X},\alpha) : \hat{X} \to X \text{ log blow-up}, \alpha \in CH^*(\hat{X})\} / (\hat{X},\alpha) \sim (\hat{\hat{X}}, \hat{\pi}^*\alpha)$ for $\hat{\hat{X}} \xrightarrow{\hat{\pi}} \hat{X} \to X$.

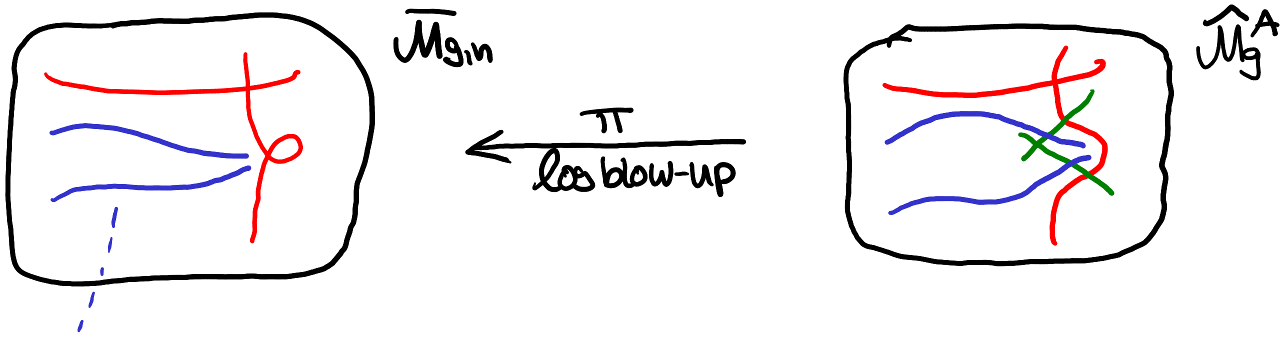
- $\log CH^*(X,D)$ is $\mathbb{Q}$-algebra,

$$\begin{tikzcd} CH^*(X) \arrow[r, hook] & \log CH^*(X,D) \\ \alpha \arrow[r, mapsto] & {[(X,\alpha)]} \end{tikzcd}$$

- $$\begin{tikzcd} \log CH^*(X,D) \arrow[r, two heads] & CH^*(X) \\ {[(\hat{X} \xrightarrow{\pi} X, \alpha)]} \arrow[r, mapsto] & \pi_*\alpha. \end{tikzcd}$$ $\mathbb{Q}$-linear

## Applications & History

- [HPS] defined the logarithmic double ramification cycle



$\mathrm{DR}_g^\circ(A) = \{(C,P_1,\ldots,P_n) : \mathcal{O}_C(\sum a_i P_i) \cong \mathcal{O}_C\}$ $\widehat{\mathrm{DR}}_g(A) \in CH^g(\hat{\mathcal{M}}_g^A)$

[Holmes]

$A = (a_1,\ldots,a_n) \in \mathbb{Z}^n$
with $\sum a_i = 0$

smooth

$$\begin{tikzcd} \rightsquigarrow \log \mathrm{DR}_g(A) \arrow[d, "\pi_*"] & = [(\hat{\mathcal{M}}_g^A, \widehat{\mathrm{DR}}_g(A))] \in \log CH^g(\overline{\mathcal{M}}_{g,n}) \\ \mathrm{DR}_g(A) & \in CH^g(\overline{\mathcal{M}}_{g,n}). \end{tikzcd}$$

- [HPS]

$$\log DR_g(A) \cdot \log DR_g(B) = \log DR_g(A) \cdot \underset{\uparrow}{\overset{\log CH^{2g}(\overline{\mathcal{M}}_{g,n}) \atop \in}{\log DR_g(A+B)}}$$

$\boxed{\text{False for } DR_g}$

**Idea** $\mathcal{O}_C(\sum a_i p_i) \cong \mathcal{O}_C$ $\quad$ $\mathcal{O}_C(\sum b_i p_i) \cong \mathcal{O}_C$ $\iff$ $\mathcal{O}_C(\sum a_i p_i) \cong \mathcal{O}_C$ $\quad$ $\mathcal{O}_C(\sum (a_i+b_i) p_i) \cong \mathcal{O}_C$

- [Molcho - Pandharipande - S.]

$\boxed{\text{False for } divCH^*}$

$$DR_g(A) \in \underset{\uparrow}{div\log} CH^*(\overline{\mathcal{M}}_{g,n})$$

sub-$\mathbb{Q}$-algebra of $\log CH^*$ gen. by $\log CH^1$.

__Conjecture__ [MPS] $\log DR_g(A) \in div\log CH^*(\overline{\mathcal{M}}_{g,n})$

↳ proven by [Molcho-Ranganathan, Holmes-Schwarz]

- [Cavalieri - Markwig - Ranganathan]

$\boxed{\text{Unknown for } DR_g}$

$$\text{Double Hurwitz Number}_g(A) = \int_{\widehat{\mathcal{M}}_g^A} \log DR_g(A) \cdot \underset{\uparrow}{br_{g,A}}.$$

generalized w/ descendant insert. in [C-M-S '24]

- [Holmes - Molcho - Pandharipande - Pixton - S.] calculate $\log DR_g(A)$ in terms of log-tautological classes on $\overline{\mathcal{M}}_{g,n}$.

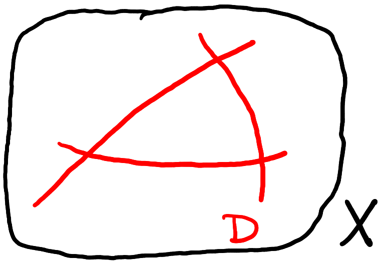
$\boxed{Q2}$ (PRSS)

What are log-tautological classes?

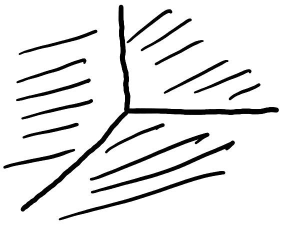
## §1 Cone stacks & Artin fans

$\mathbb{G}_m \simeq \mathbb{C}^*$

**Case 1** $X$ smooth toric variety with torus $T \cong \mathbb{G}_m^n \overset{\text{open}}{\subseteq} X$, $D = X \setminus T$

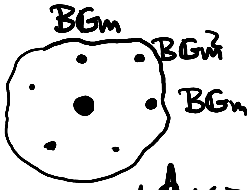


$\rightsquigarrow$



$\Sigma_X$ fan

$\downarrow q_X$



$\mathcal{A}_X = [X/T]$

**Thm** (Brion) $\exists$ isom.

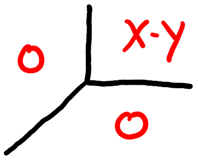
Strict piecew. Polynomials on $\Sigma_X$

$$\begin{tikzcd}
SPP^*(\Sigma_X) \arrow[d, "\Phi", "\cong"'] \arrow[dr] & \\
CH^*(\mathcal{A}_X) \arrow[r, "q_X^*"] & CH^*(X)
\end{tikzcd}$$

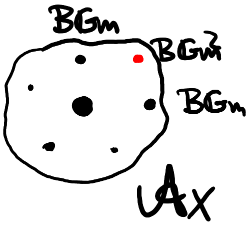
surjective with kernel $= \langle L(\Sigma) \rangle$

linear fct. on $\Sigma$

**Exa**



$= f \Rightarrow \Phi(f) = [BG_m^2]$



$\mathcal{A}_X$

**Moreover**

log blow-ups of $X$ (or $\mathcal{A}_X$)

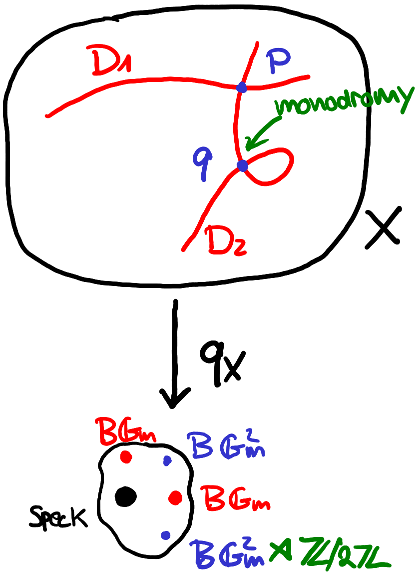
$\widehat{=}$ subdivisions of $\Sigma_X$

## Case 2 $(X,D)$ smooth nc pair

Idea · étale locally around $P \in (X,D)$: $D$ looks toric $\rightsquigarrow \Sigma_P$

· étale patches glue to $(X,D)$ $\rightsquigarrow$ $\Sigma_P$ glue to $\Sigma_X$

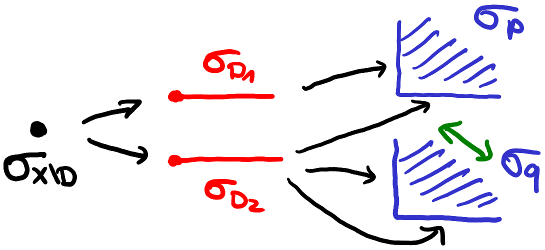
$$\begin{tikzcd}(X,D) \arrow[d] \\ (\mathcal{A}_X,\mathcal{D})\end{tikzcd}$$



$\mathcal{A}_{(X,D)}$ Artin fan

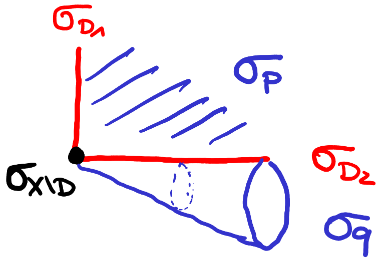
[Abramovich–Chen–Marcus –Ulirsch–Wise]

$\rightsquigarrow$



$\Sigma_{(X,D)}$: cone stack

[Cavalieri–Chan–Ulirsch–Wise]



### Moreover

Subdivisions $\hat{\Sigma} \to \Sigma_{(X,D)}$ $\hat{=}$ log blow-ups

$$\begin{tikzcd}(\hat{X},\hat{D}) \arrow[r] \arrow[d, "q_{\hat{X}}"] \arrow[dr, phantom, "\square"] & (X,D) \arrow[d, "q_X"] \\ (\hat{\mathcal{A}},\hat{\mathcal{D}}) \arrow[r] & (\mathcal{A}_X,\mathcal{D})\end{tikzcd}$$

**Summary**

$$\begin{tikzcd}
(X,D) \arrow[r, squiggly] \arrow[d, "q_X"] & \Sigma_{(X,D)} \text{ cone stack} \\
(\mathcal{A}_X,\mathcal{D}) \text{ Artin fan} &
\end{tikzcd}$$

**Thm [MPS]** ∃ isom.

$$\begin{tikzcd}
sPP^*(\Sigma_{(X,D)}) \arrow[r, "\Phi", "\sim"'] & CH^*(\mathcal{A}_X) \arrow[r, "q_X^*"] & CH^*(X) \\
PP^*(\Sigma_{(X,D)}) \arrow[r, "\Phi^{\log}", "\sim"'] & \log CH^*(\mathcal{A}_X,\mathcal{D}) \arrow[r, "q_X^*"] & \log CH^*(X,D)
\end{tikzcd}$$

functions $\Sigma_{(X,D)} \to \mathbb{R}$ compatible with face maps & polynomial (on all cones / on some subdivision $\hat{\Sigma} \to \Sigma_{(X,D)}$)
sPP* PP*

↑ both $q_X^*$ no longer surjective

Image of $q_X^* \circ \Phi^{(\log)}$ : normally decorated strata classes

↗ classes of strata closures in $X$ (or $\tilde{X}$) decorated by Chern classes of normal bundles.

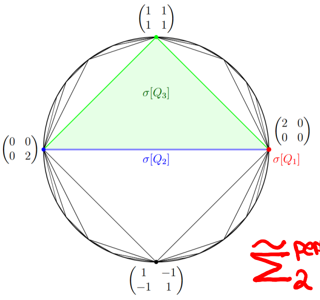
$\rightsquigarrow \boxed{Q1}$

**Application** Intersection theory on $\overline{\mathcal{A}}_g$

Toroidal compactifications $\overline{\mathcal{A}}_g$ of $\mathcal{A}_g$

⇕

Admissible decompositions $\tilde{\Sigma}$ of



$\tilde{\Sigma}_2$ perfect cone

$$\Omega_g^{rt} = \{ Q \in \mathrm{Sym}_{g\times g}(\mathbb{R}) : Q \geq 0,\ \ker(Q) \subseteq \mathbb{R}^g \text{ rational} \}$$

$\rightsquigarrow$ cone stack of $\overline{\mathcal{A}}_g^{\tilde{\Sigma}} = \tilde{\Sigma} / GL_g(\mathbb{Z})$

**Conj** (Pixton) Constructs explicit $P \in PP^*(\Omega_g^{rt})^{GL_g(\mathbb{Z})}$

s.t. $q_{\overline{\mathcal{A}}}^* \circ \Phi^{\log}(P) = \lambda_g \in \log CH^g(\overline{\mathcal{A}}_g)$.

## §2 Applications to moduli spaces of curves

**Def** The small log-tautological ring $\log R^*_{sm}(\overline{\mathcal{M}}_{g,n})$ is the $\mathbb{Q}$-subalgebra of $\log CH^*(\overline{\mathcal{M}}_{g,n})$ gen. by

- $R^*(\overline{\mathcal{M}}_{g,n}) \subseteq CH^*(\overline{\mathcal{M}}_{g,n}) \subseteq \log CH^*(\overline{\mathcal{M}}_{g,n})$ taut. classes
- $\mathrm{im}\left(\Phi^{\log} : PP^*(\mathcal{M}^{trop}_{g,n}) \longrightarrow \log CH^*(\overline{\mathcal{M}}_{g,n})\right)$

$= \Sigma(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$ moduli space of tropical curves

**Thm** (HMPPS)

$$\log DR_g(A) = \left[\exp\left(\eta + \Phi^{\log}(f_L)\right)\cdot \Phi^{\log}(f_P)\right]_g \in \log R^g_{sm}(\overline{\mathcal{M}}_{g,n})$$

$\eta = \sum \frac{a_i^2}{2}\psi_i \in R^1(\overline{\mathcal{M}}_{g,n})$ $f_L, f_P \in PP^*(\mathcal{M}^{trop}_{g,n})$ codim g part.

**Thm** (PRSS)

$$\Phi^{\log} : PP^*(\mathcal{M}^{trop}_{0,n}) \twoheadrightarrow \log CH^*(\overline{\mathcal{M}}_{0,n})$$

is surjective, kernel = gen. by $WDVV^{PP}_{0,n}$

Piecew. linear fcts. on $\mathcal{M}^{trop}_{0,n}$ mapping to WDVV-rel's as norm. decorated strata classes.

**Idea of proof**

[Kapranov] constructs $\underbrace{\overline{\mathcal{M}}_{0,n} \xhookrightarrow{i} X_n}$ smooth q.proj. toric variety

$i$ induces $\Sigma_{\overline{\mathcal{M}}_{0,n}} \cong \Sigma_{X_n}$, $\Sigma_{\overline{\mathcal{M}}_{0,n}} = \mathcal{M}^{trop}_{0,n}$

$i^*$ induces isom. of $CH^*$ (on all strata)

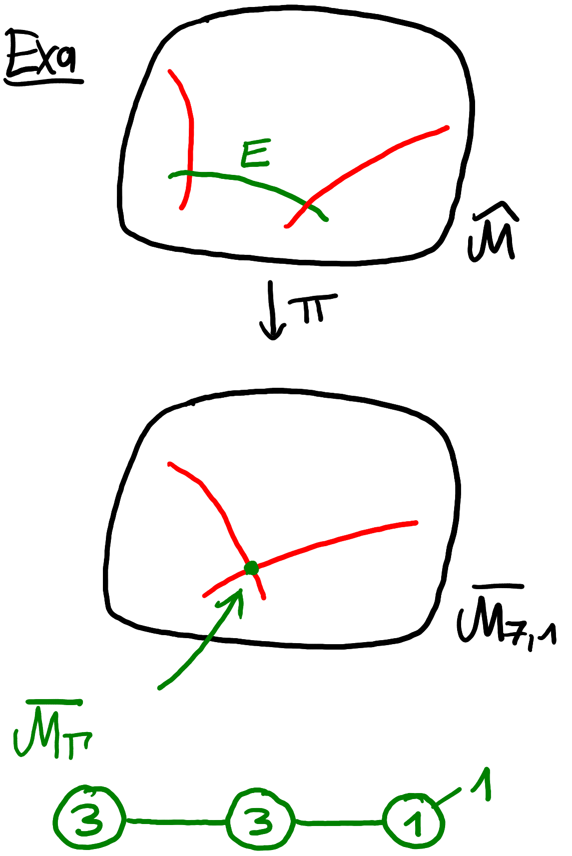
$$\Rightarrow \log CH^*(\overline{\mathcal{M}}_{0,n}) = \log CH^*(X_{0,n}) \overset{\text{Brion}}{=} PP^*(\underbrace{\Sigma_{X_n}}_{= \mathcal{M}^{trop}_{0,n}}) / (\underbrace{L(\Sigma_{X_n})}_{\overset{\text{Check}}{=} WDVV^{PP}_{0,n}}) \quad \square$$

# §3 Larger log-tautological rings

**Problem**

Some natural classes are missing from $\log R^*_{sm}(\overline{\mathcal{M}}_{g,n})$

**Exa**



$$\begin{tikzcd} E \arrow[r, hook, "i"] \arrow[d, "\pi_E"'] & \hat{\mathcal{M}} \\ \overline{\mathcal{M}}_\Gamma & \end{tikzcd}$$

$\rightsquigarrow$ $\alpha = \kappa_1 \otimes 1 \otimes 1$

$\rightsquigarrow [\hat{\mathcal{M}}, i_* \pi_E^* \alpha] \in \log CH^2(\overline{\mathcal{M}}_{7,1})$
looks tautological,
but not in $\log R^*_{sm}(\overline{\mathcal{M}}_{7,1})$.

**Idea** $(X,D)$ snc (for simplicity)

Let $\hat{X} \xrightarrow{\pi} X$ corr. to

$$\begin{tikzcd} \hat{\Sigma} \arrow[r] & \Sigma_{(X,D)} \\ \hat{\sigma} \arrow[u, phantom, "\in" sloped] \arrow[r, mapsto] & \sigma \arrow[u, phantom, "\in" sloped] \end{tikzcd}$$

$$\begin{tikzcd} \overline{S}_{\hat{\sigma}} \arrow[r, hook, "i_{\hat{\sigma}}"] \arrow[d, "p_{\hat{\sigma}}"] & \hat{X} \arrow[d, "\pi"] \\ \overline{S}_\sigma \arrow[r, hook] & X \end{tikzcd}$$

strata closures

$$\begin{tikzcd} & \overline{S}_{\hat{\sigma}} \arrow[dl, "p_{\hat{\sigma}}"'] \arrow[r, hook, "i_{\hat{\sigma}}"] \arrow[d, "q_{\hat{\sigma}}"] & \hat{X} \arrow[d, "q_{\hat{X}}"] \\ \overline{S}_\sigma & \mathcal{B}_{\hat{\sigma}} \arrow[r, hook, "\text{closed}"] & \mathcal{A}_{\hat{X}} \end{tikzcd}$$

**Thm** (PRSS)

$\exists$ isom.

$$\begin{array}{c} \Psi : \{ f \in sPP^*(\hat{\Sigma}) : f \equiv 0 \text{ outside } \mathrm{Star}_{\hat{\sigma}} \} \\ \downarrow \cong \\ CH_*(\mathcal{B}_{\hat{\sigma}}) \end{array}$$

**Def** $\log R^*(X, D)$ is $\mathbb{Q}$-vector space gen. by

$$[\hat\sigma, f, \alpha] := \big[\big(\hat X, (i_{\hat\sigma})_*\big(p_{\hat\sigma}^*\alpha \cap q_{\hat\sigma}^*\Psi(f)\big)\big)\big]$$

$\hat\sigma$ cone in some $\hat\Sigma \to \Sigma$

$f$ as above

$\alpha \in CH^*(\overline{S}_\sigma)$ decoration

$\boxed{\text{CHOICE}}$

$\in \log CH^*(X, D)$

Then

- allowing $\alpha =$ poly in $\kappa, \psi$-classes in $CH^*(\overline{\mathcal{M}}_\Gamma)$
  $\rightsquigarrow$ class from above is in $\log R^*(\overline{\mathcal{M}}_{g,n})$
- $\log R^*(X, D)$ completely determined by $(CH^*(\overline{S}_\sigma))_{\sigma \in \Sigma_{(X,D)}}$ & maps between them.
- $\alpha \in CH^*(\overline{S}_\sigma)$ allowed to be arb. elem. of $CH^*(\overline{S}_\sigma)$

$\rightsquigarrow \log R^*_{CH}(X, D) = \log CH^*(X, D)$

$\rightsquigarrow [\hat\sigma, f, \alpha]$ give additive generating set.

Thank you for your attention!