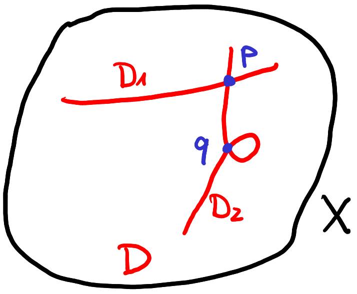
Piecewise polynomials & logarithmic tautological rings

§0 Motivation

(X,D) smooth space with normal crossings divisor (smooth log smooth)



E.g. $(X = \overline{\mathcal{M}}_{g,n}$, $D = \partial \overline{\mathcal{M}}_{g,n})$

mod. space of stable curves

locus of singular curves

$\rightsquigarrow$ stratification $X = \bigsqcup_\sigma S_\sigma$ into loc. closed $S_\sigma \subseteq X$.

$\boxed{Q1}$ How to intersect classes $[\overline{S}_\sigma] \in CH^*(X)$?

$\rightsquigarrow$ nice combinatorial formalism ?

E.g. $[D_1]\cdot[D_2] = [p] \in CH^2(X)$ above.

Next problem sometimes $U = X \setminus D$ is canonical, but it's (partial) compactification X is not !

Exa $U = \mathcal{A}_g$: moduli space of princ. polarized abel. var of dim g

$\rightsquigarrow X = \overline{\mathcal{A}}_g$: different birat'l models $(\overline{\mathcal{A}}_g^{PC}, \overline{\mathcal{A}}_g^{Vor}, \ldots)$

In good cases:

$$\begin{tikzcd} & \exists\, \hat{X} \arrow[dl, dashed] \arrow[dr, dashed] & \\ X_1 & & X_2 \\ & U \arrow[ul, phantom, "\supseteq"] \arrow[ur, phantom, "\subseteq"] & \end{tikzcd}$$

any two X_1, X_2 receive map from common space $\hat{X}$ and $\hat{X} \to X_i$ is a log blow-up

think: iterated blow-up of smooth strata closures

Def (Holmes–Pixton–S.)

(X,D) smooth nc pair.

$$\rightsquigarrow \log CH^*(X,D) := \varinjlim_{\substack{(\hat{X},\hat{D}) \to (X,D) \\ \text{log blow-up}}} CH^*(\hat{X})$$

trans. maps = pullb. $\hat{\pi}^*$

$\hat{\hat{X}} \xrightarrow{\hat{\pi}} \hat{X} \longrightarrow X$

logarithmic Chow ring

Basic properties

- $\log CH^*(X,D) = \{(\hat{X},\alpha) : \hat{X} \to X \text{ log blow-up}, \alpha \in CH^*(\hat{X})\} / (\hat{X},\alpha) \sim (\hat{\hat{X}}, \pi^*\alpha)$ for $\hat{\hat{X}} \xrightarrow{\pi} \hat{X} \to X$.

- $\log CH^*(X,D)$ is $\mathbb{Q}$-algebra,

$$\begin{aligned} CH^*(X) &\hookrightarrow \log CH^*(X,D) \\ \alpha &\longmapsto [(X,\alpha)] \end{aligned}$$

- $\log CH^*(X,D) \twoheadrightarrow CH^*(X)$ $\quad$ $\mathbb{Q}$-linear

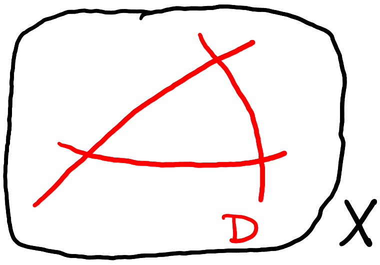
$$[(\hat{X} \xrightarrow{\pi} X, \alpha)] \longmapsto \pi_*\alpha.$$

$\boxed{\text{Q2}}$ How to construct natural classes in $\log CH^*(X,D)$ & work with them?

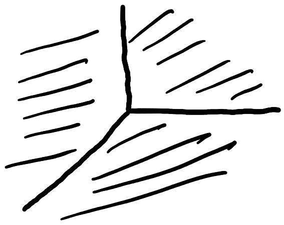
§1 Cone stacks & Artin fans

<u>Case 1</u> X smooth toric variety with torus $T \cong \mathbb{G}_m^n \overset{\text{open}}{\subseteq} X$, $D = X \setminus T$

$\mathbb{G}_m = \mathbb{C}^*$

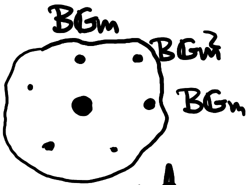


$\rightsquigarrow$



Σ_X fan

$\downarrow q_X$



$\mathcal{A}_X = [X/T]$

<u>Thm</u> (Brion) $\exists$ isom.

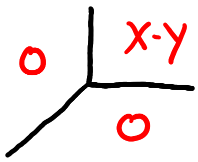
$$\begin{tikzcd} SPP^*(\Sigma_X) \arrow[d, "\cong"', "\Phi"] \arrow[dr] & \\ CH^*(\mathcal{A}_X) \arrow[r, "q_X^*"] & CH^*(X) \end{tikzcd}$$

strict piecew. polynomials on Σ_X

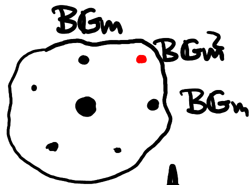
surjective with kernel $= \langle L(\Sigma) \rangle$

linear fct. on Σ

<u>Exa</u>



$= f \implies \Phi(f) = [BG_m^2]$



$\mathcal{A}_X$

<u>Moreover</u>

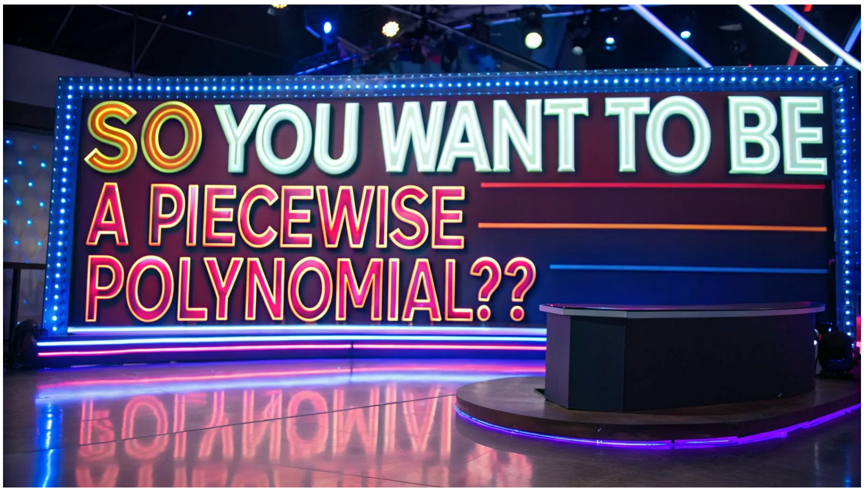
log blow-ups of X (or $\mathcal{A}_X$)

$\widehat{=}$ subdivisions of Σ_X

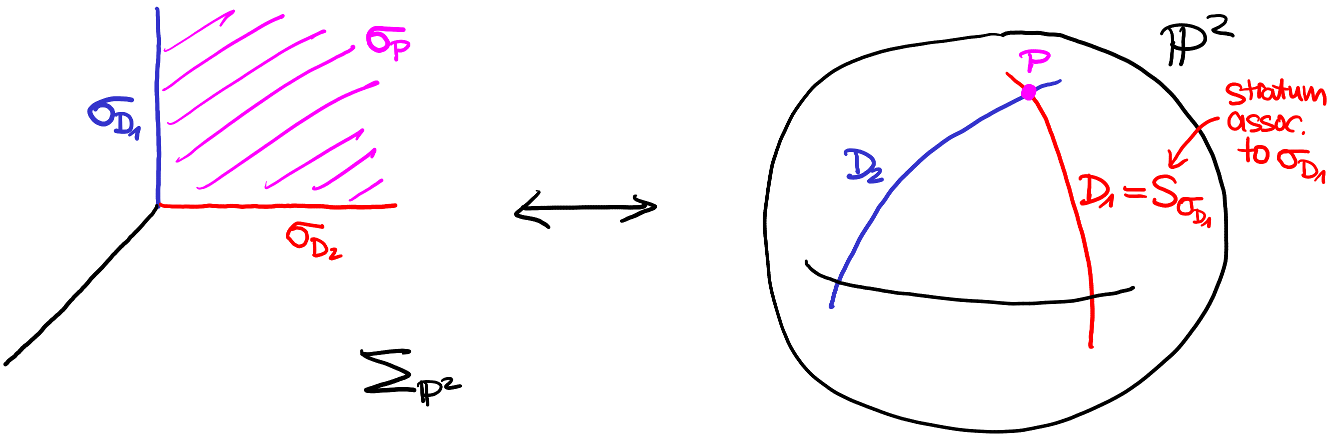
<u>Cor</u> $\log CH^*(X) \cong PP^*(\Sigma)/\langle L(\Sigma)\rangle = \varinjlim_{\hat{\Sigma} \to \Sigma} sPP^*(\hat{\Sigma})/\langle L(\Sigma)\rangle$

strict piecew. polyn. on <u>some</u> subdivision $\hat{\Sigma} \to \Sigma$

Interlude

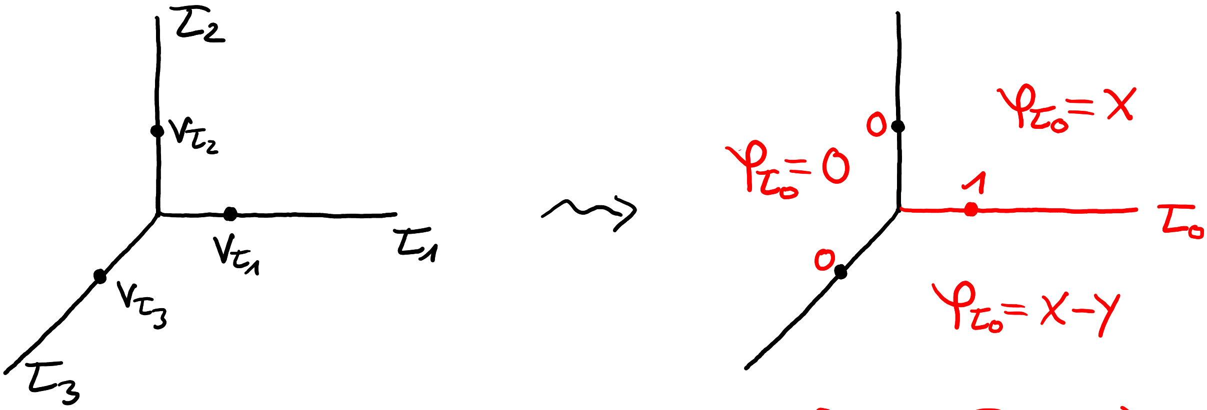


Correspondence between cones & strata



Piecewise polynomials associated to cones / strata

Σ in $\mathbb{R}^n$ simplicial, $\tau \in \Sigma$ ray $\rightsquigarrow$ $v_\tau \in N = \mathbb{Z}^n \subseteq \mathbb{R}^n$
primitive ray generator



$\forall \tau_0 \ \exists! \ \varphi_{\tau_0} \in SPP^1(\Sigma)$
with

$$\varphi_{\tau_0}(v_\tau) = \begin{cases} 1 & , \tau = \tau_0 \\ 0 & , \tau \neq \tau_0 \end{cases}$$

<u>Exercise</u> Calculate $\varphi_{\tau_2}, \varphi_{\tau_3}$ above.

<u>Higher dimensions</u>

$\sigma \in \Sigma(d)$ cone of dim. d, rays $\sigma(1) = \{\tau_1, \dots, \tau_d\}$

$$\Rightarrow \quad \varphi_\sigma := \prod_{\tau \in \sigma(1)} \varphi_\tau = \varphi_{\tau_1} \cdots \varphi_{\tau_d} \in SPP^d(\Sigma).$$

<u>Exercise</u> Calculate φ_σ for all $\sigma \in \Sigma$ above.

<u>Thm</u> (Brion) $\exists$ isom.

$$\begin{tikzcd} SPP^*(\Sigma_X) \arrow[d, "\cong"', "\Phi"] \arrow[dr] & \\ CH^*(\mathcal{A}_X) \arrow[r, "q_X^*"] & CH^*(X) \end{tikzcd}$$

strict piecew. polynomials on Σ_X

surjective with kernel $= \langle L(\Sigma) \rangle$

linear fct. on Σ

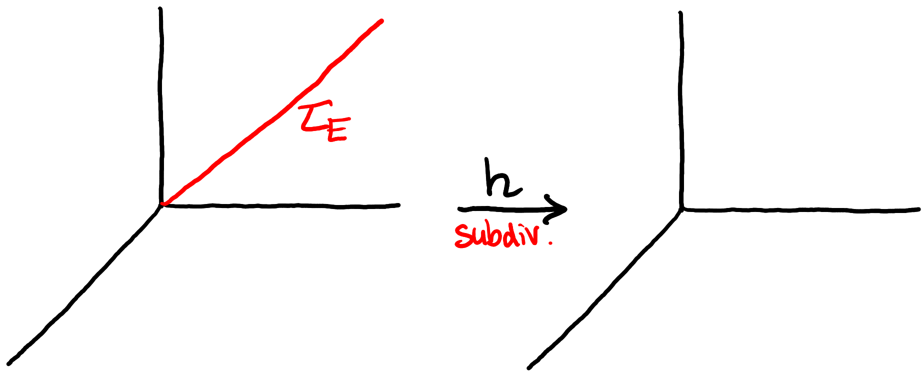
$$\rightsquigarrow \quad (q_X^* \circ \Phi)(\varphi_\sigma) = [\overline{S}_\sigma] \in CH^d(X)$$

class of strata closure ass. to $\sigma \in \Sigma$

<u>Fact</u> $\{\varphi_\tau : \tau \in \Sigma(1)\}$ generates $SPP^*(\Sigma)$ as $\mathbb{Q}$-algebra

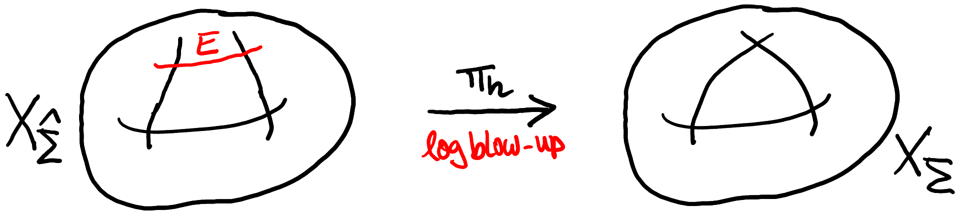
$\Rightarrow$ $(q_X^* \circ \Phi)$ uniquely determined.

Proper birational pushforwards of piecewise polynomials



(★)

$h: \hat{\Sigma} \to \Sigma$ subdivision



$\pi_h: X_{\hat{\Sigma}} \to X_\Sigma$ log blow-up (birat'l, proper)

$$\rightsquigarrow \quad \begin{tikzcd} CH^*(X_{\hat{\Sigma}}) \arrow[r, "(\pi_h)_*"] & CH^*(X_\Sigma) \\ sPP^*(\hat{\Sigma}) \arrow[u, "\Phi_{\hat{\Sigma}}"] \arrow[r, "\exists h_*"] & sPP^*(\Sigma) \arrow[u, "\Phi_\Sigma"'] \end{tikzcd} \qquad (★★)$$

allows to compute $\pi_*: \log CH^*(X) \to CH^*(X)$

Thm (Brion) $\hat{\Sigma} \xrightarrow{h} \Sigma$ subdivision of simplicial fans
$f \in sPP^*(\hat{\Sigma}) \Rightarrow \exists g = h_* f \in sPP^*(\Sigma)$ s.t. (★★) commutes:

$$g|_\sigma = \varphi_\sigma|_\sigma \cdot \sum_{\substack{\hat{\sigma} \in h^{-1}(\sigma) \\ \dim(\hat{\sigma}) = \dim(\sigma)}} \left(\frac{1}{\varphi_{\hat{\sigma}}|_{\hat{\sigma}}} \right) \cdot f|_{\hat{\sigma}} \in \mathbb{Q}[x_1, \dots, x_n]$$

$\forall \sigma \in \Sigma(n)$

<u>Note</u> A priori, the RHS is a rat'l function in $x_1, \dots, x_n$
but a posteriori all numerators cancel, giving a polynom.

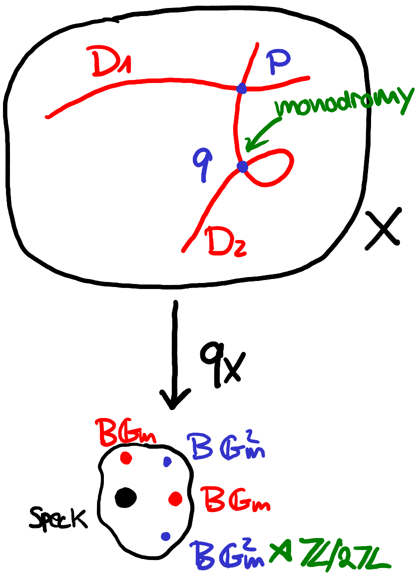
<u>Exercise</u> For subdiv. (★), calculate $h_*(\varphi_{\sigma_E}^2) \in sPP^2(\Sigma)$.

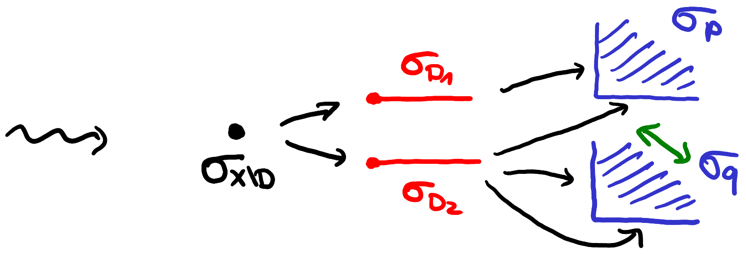
<u>Question</u> (j/w Fousi, Iribar Lopez, Molcho)
Does a canonical h_* exist for all proper maps $\pi_h: X_{\hat{\Sigma}} \to X_\Sigma$, st. (i) (★★) commutes, (ii) h_* map of $sPP^*(\Sigma)$-modules
(iii) $\mathrm{supp}(f) \subseteq \mathrm{Star}_{\hat{\sigma}} \hat{\Sigma} \Rightarrow \mathrm{supp}(h_* f) \subseteq \mathrm{Star}_{h(\hat{\sigma})}(\Sigma)$?

Case 2 (X,D) smooth nc pair

Idea
- étale locally around $P \in (X,D)$: D looks toric $\rightsquigarrow \Sigma_P$
- étale patches glue to $(X,D) \rightsquigarrow \Sigma_P$ glue to Σ_X

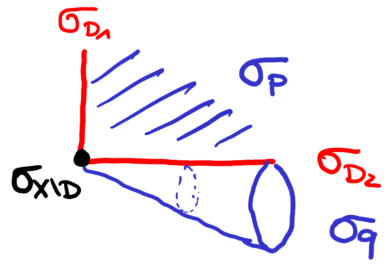
$$\begin{array}{c}(X,D)\\ \downarrow \\ (\mathcal{A}_X,\mathcal{D})\end{array}$$





$\Sigma_{(X,D)}$: cone stack

[Cavalieri-Chan-Ulirsch-Wise]



$\mathcal{A}_{(X,D)}$ Artin fan

[Abramovich-Chen-Marcus-Ulirsch-Wise]

Moreover

Subdivisions $\hat{\Sigma} \to \Sigma_{(X,D)}$ $\hat{=}$ log blow-ups

$$\begin{tikzcd}
(\hat{X},\hat{D}) \arrow[r] \arrow[d, "q_{\hat{X}}"] \arrow[dr, phantom, "\square"] & (X,D) \arrow[d, "q_X"] \\
(\hat{\mathcal{A}},\hat{\mathcal{D}}) \arrow[r] & (\mathcal{A}_X,\mathcal{D})
\end{tikzcd}$$

Summary

$$\begin{array}{ccc} (X,D) & \rightsquigarrow & \Sigma_{(X,D)} \text{ cone stack} \\ \big\downarrow q_X & & \\ (\mathcal{A}_X, \mathcal{D}) & & \text{Artin fan} \end{array}$$

Thm [MPS]
$\exists$ isom.

$$\mathrm{sPP}^*(\Sigma_{(X,D)}) \xrightarrow[\sim]{\Phi} CH^*(\mathcal{A}_X) \xrightarrow{q_X^*} CH^*(X)$$

$$\mathrm{PP}^*(\Sigma_{(X,D)}) \xrightarrow[\sim]{\Phi^{\log}} \log CH^*(\mathcal{A}_X, \mathcal{D}) \xrightarrow{q_X^*} \log CH^*(X,D)$$

functions $\Sigma_{(X,D)} \to \mathbb{R}$ compatible with face maps & polynomial (on all cones / on some subdivision $\hat{\Sigma} \to \Sigma_{(X,D)}$)
SPP*   PP*

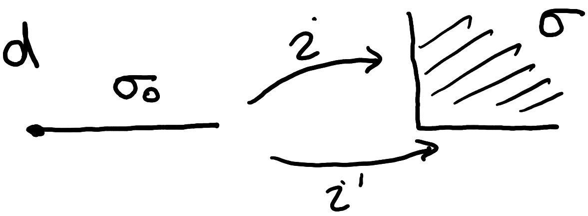
both q_X^* no longer surjective

Image of $q_X^* \circ \Phi^{(\log)}$: normally decorated strata classes

classes of strata closures in X (or $\tilde{X}$) decorated by Chern classes of normal bundles. $\rightsquigarrow$ $\boxed{Q1}$

How does this map work?

Σ simplicial, $\sigma_0 \in \Sigma$, $\dim \sigma_0 = d$

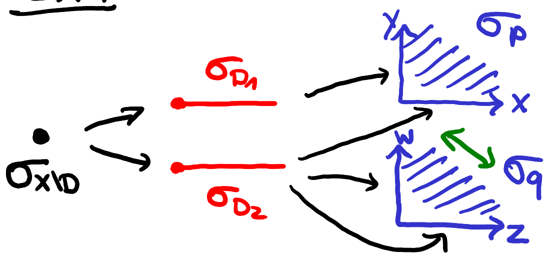


Want $\varphi_{\sigma_0} \in \mathrm{sPP}^d(\Sigma)$
s.t. $(q_X^* \circ \Phi)(\varphi_{\sigma_0}) = [\overline{S}_{\sigma_0}]$

Given $\sigma \in \Sigma$:

$$\varphi_{\sigma_0}\big|_{\sigma} := \frac{1}{|\mathrm{Aut}(\sigma_0)|} \cdot \sum_{\underbrace{i : \sigma_0 \to \sigma}_{\text{face inclusions in } \Sigma}} \underbrace{\varphi_{i(\sigma_0) \subseteq \sigma}}_{= \prod_{\tau \in \sigma_0(1)} x_{i(\tau)}}$$

Exa



$\varphi_{\sigma_{x\times D}}$	$\varphi_{\sigma_{D_1}}$	$\varphi_{\sigma_{D_2}}$	φ_{σ_p}	φ_{σ_q}
1	y	x	$x\cdot y$	0
1	0	$w+z$	0	$\frac{1}{2}\cdot(wz+zw) = wz$

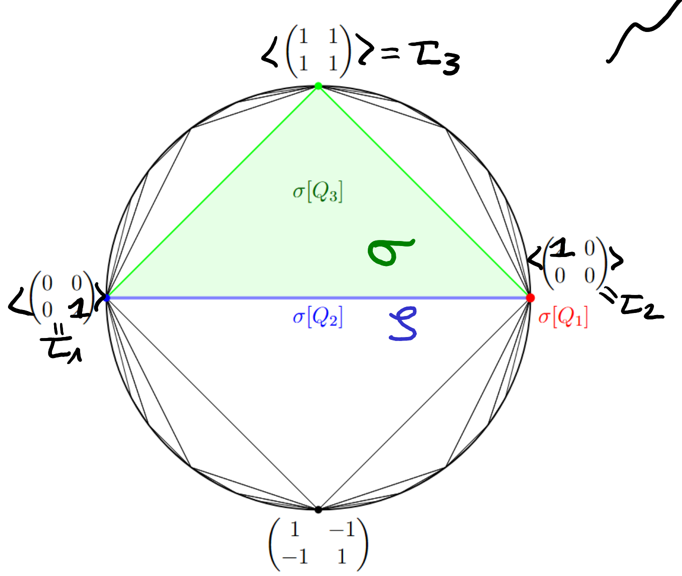
Application (Piecewise) polynomials on $\Sigma = \mathcal{A}_g^{trop}$

Recall:

$$\Omega_g^{rt} = \{Q \in Sym_{g\times g}(\mathbb{R}) : Q \geq 0,\ Ker(Q)\ rat'l\}$$

admissible decompos. Σ of Ω_g^{rt} $\triangleq$ toroidal comp. $\overline{\mathcal{A}}_g^{\Sigma}$ of $\mathcal{A}_g$

Exa Σ_2^{PC} for $g=2$ coordinates: $Q = \begin{pmatrix} a & b \\ b & c\end{pmatrix}$



Exercise

What is

$$\varphi_{\varsigma}|_{\sigma} \in \mathbb{Q}[a,b,c]\ ?$$

Thm (Feusi, Iribar-Lopez)

$$\Phi(\det Q) = cst. \cdot [\overline{\mathcal{A}}^{rk=g}] \in CH^g(\overline{\mathcal{A}}_g^{PC})$$

maybe 1 or $1/2^g$

locus where torus rank $= g$

§3 Log decorated strata classes

Q Log tautological classes on $\overline{\mathcal{M}}_{g,n}$?

Def The <u>small log-tautological ring</u> $\log R^*_{sm}(\overline{\mathcal{M}}_{g,n})$ is the $\mathbb{Q}$-subalgebra of $\log CH^*(\overline{\mathcal{M}}_{g,n})$ gen. by

- $R^*(\overline{\mathcal{M}}_{g,n}) \subseteq CH^*(\overline{\mathcal{M}}_{g,n}) \subseteq \log CH^*(\overline{\mathcal{M}}_{g,n})$ taut. classes
- $\mathrm{im}\left(\Phi^{\log} : PP^*(\mathcal{M}^{trop}_{g,n}) \longrightarrow \log CH^*(\overline{\mathcal{M}}_{g,n})\right)$

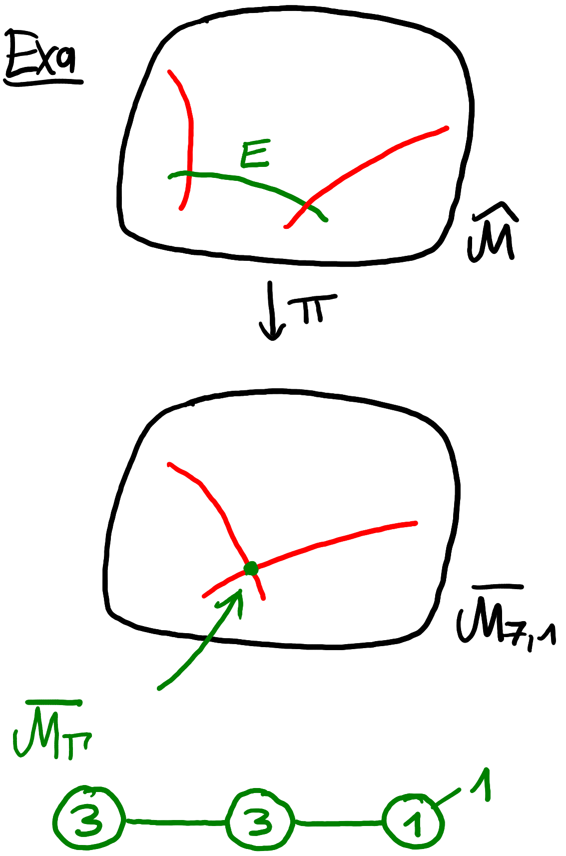
$\mathcal{M}^{trop}_{g,n} = \Sigma(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$ moduli space of tropical curves

$\rightsquigarrow \log DR_g(A) \in \log R^*_{sm}(\overline{\mathcal{M}}_{g,n})$ [HMPPS]

<u>Problem</u>

Some natural classes are missing from $\log R^*_{sm}(\overline{\mathcal{M}}_{g,n})$

<u>Exa</u>



$\rightsquigarrow$
$$\begin{array}{ccc} E & \overset{i}{\hookrightarrow} & \widehat{\mathcal{M}} \\ {\scriptstyle \pi_E}\downarrow & & \\ \overline{\mathcal{M}}_\Gamma & & \end{array} \qquad \alpha = \kappa_1 \otimes 1 \otimes 1$$

$\rightsquigarrow [\widehat{\mathcal{M}}, i_* \pi_E^* \alpha] \in \log CH^2(\overline{\mathcal{M}}_{7,1})$ looks tautological, but <u>not</u> in $\log R^*_{sm}(\overline{\mathcal{M}}_{7,1})$.

<u>Idea</u> (X,D) snc (for simplicity)

Let $\hat{X} \xrightarrow{\pi} X$ corr. to $\hat{\Sigma} \longrightarrow \Sigma_{(X,D)}$

$$\hat{\sigma} \longmapsto \sigma \quad (\hat{\sigma} \in \hat{\Sigma},\ \sigma \in \Sigma_{(X,D)})$$

$$\begin{tikzcd} \overline{S}_{\hat{\sigma}} \arrow[r, hook, "i_{\hat{\sigma}}"] \arrow[d, "p_{\hat{\sigma}}"] & \hat{X} \arrow[d, "\pi"] \\ \overline{S}_{\sigma} \arrow[r, hook] & X \end{tikzcd}$$

strata closures

$$\begin{tikzcd} & \overline{S}_{\hat{\sigma}} \arrow[dl, "p_{\hat{\sigma}}"'] \arrow[r, hook, "i_{\hat{\sigma}}"] \arrow[d, "q_{\hat{\sigma}}"] \arrow[dr, phantom, "\square"] & \hat{X} \arrow[d, "q_{\hat{X}}"] \\ \overline{S}_{\sigma} & B_{\hat{\sigma}} \arrow[r, hook, "\text{closed}"] & A_{\hat{X}} \end{tikzcd}$$

<u>Thm</u> (PRSS)
$\exists$ isom.

$$\Psi : \{ f \in sPP^*(\hat{\Sigma}) : f \equiv 0 \text{ outside } \mathrm{Star}_{\hat{\sigma}} \} \xrightarrow{\ \cong\ } CH_*(B_{\hat{\sigma}})$$

<u>Def</u> $\log R^*(X,D)$ is $\mathbb{Q}$-vector space gen. by

$$[\hat{\sigma}, f, \alpha] := [(\hat{X}, (i_{\hat{\sigma}})_*(p_{\hat{\sigma}}^* \alpha \cap q_{\hat{\sigma}}^* \Psi(f)))] \in \log CH^*(X,D)$$

$\hat{\sigma}$ cone in some $\hat{\Sigma} \to \Sigma$ — f as above — $\alpha \in CH^*(\overline{S}_\sigma)$ decoration $\boxed{\text{CHOICE}}$

<u>Then</u>

- allowing α = poly in κ, ψ-classes in $CH^*(\overline{\mathcal{M}}_\Gamma)$
  $\rightsquigarrow$ class from above is in $\log R^*(\overline{\mathcal{M}}_{g,n})$

- $\log R^*(X,D)$ completely determined by $(CH^*(\overline{S}_\sigma))_{\sigma \in \Sigma(X,D)}$ & maps between them.

- $\alpha \in CH^*(\overline{S}_\sigma)$ allowed to be arb. elem. of $CH^*(\overline{S}_\sigma)$

  $\rightsquigarrow \log R^*_{CH}(X,D) = \log CH^*(X,D)$

  $\rightsquigarrow [\hat{\sigma}, f, \alpha]$ give additive generating set.

$\rightsquigarrow$ $\boxed{Q2}$

Idea for $\overline{\mathcal{A}_g}^{\Sigma}$ $\sigma \in \Sigma$ with generic (torus) rank h

$\Longrightarrow$ $\overline{S_\sigma} \xrightarrow{p_\sigma} \overline{\mathcal{A}_{g-h}}^{\tilde{\Sigma}}$ $\rightsquigarrow$ $\alpha \in p_\sigma^* \mathbb{Q}[\lambda_1, \dots, \lambda_{g-h}]$

decorations w/ Hodge classes

Thank you for your attention!