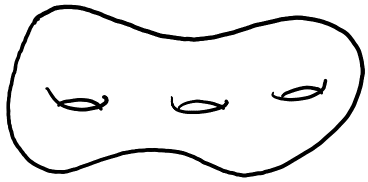
Euler characteristics of strata of K-differentials

§1 K-differentials and their moduli spaces



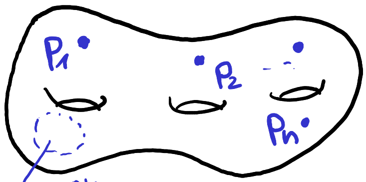
C smooth curve/$\mathbb{C}$ of genus g

$\rightsquigarrow$ $\omega_C = T_C^*$ cotangent bundle

$$\deg(\omega_C) = \int_C c_1(\omega_C) \overset{\text{Gauss-Bonnet}}{=} (-1)\cdot \chi^{top}(C)$$

$$= 2g-2$$

Def For $K \in \mathbb{Z}$, a K-differential η on C is a meromorphic section of $\omega_C^{\otimes K}$.



$\eta = f(z)\cdot dz^{\otimes K}$, f merom.

$$\underbrace{\mathrm{div}(\eta)}_{(\text{zeros})-(\text{poles})} = \sum_{i=1}^{n} m_i \cdot P_i \Rightarrow \sum_{i=1}^{n} m_i = \underbrace{K(2g-2)}_{\deg \omega_C^{\otimes K}}$$

Def Given $\mu = (m_1, \dots, m_n) \in \mathbb{Z}^n$ with $\sum_{i=1}^{n} m_i = K(2g-2)$:

$$\underbrace{\mathcal{H}_g^K(\mu)}_{\text{stratum of K-diff.}} = \left\{ (C, P_1, \dots, P_n) \in \underbrace{\mathcal{M}_{g,n}}_{\text{moduli of smooth n-pointed genus g curves}} \;\middle|\; \begin{array}{l} \exists \text{ merom. K-diff. } \eta \text{ on } C \\ \text{with } \mathrm{div}(\eta) = \sum_{i=1}^{n} m_i P_i \end{array} \right\} \subseteq \mathcal{M}_{g,n}$$

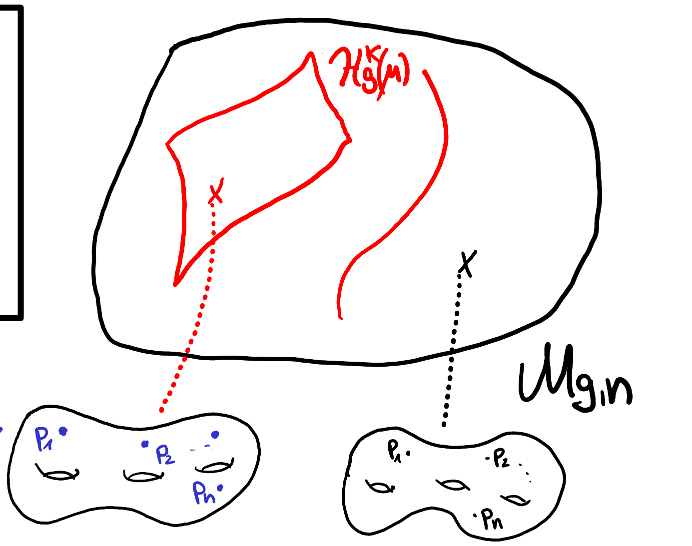
Plan of talk

§2 Basic properties of $\mathcal{H}_g^K(\mu)$

§3 Conjecture on $\chi^{orb}(\mathcal{H}_g^K(\mu))$

§4 Evidence & calculations

uniquely det. by $P_1, \dots, P_n$ up to scaling $\to \eta$



§2 A guided tour to the strata of k-differentials

<u>Basic symmetry</u>: $\mathcal{H}_g^{-k}(-\mu) = \mathcal{H}_g^{k}(\mu)$, since $\mathrm{div}(\eta^{-1}) = -\mathrm{div}(\eta)$

$\rightsquigarrow$ can restrict to $k \geq 0$ below

§2.1 Strata of 0-differentials (k=0)

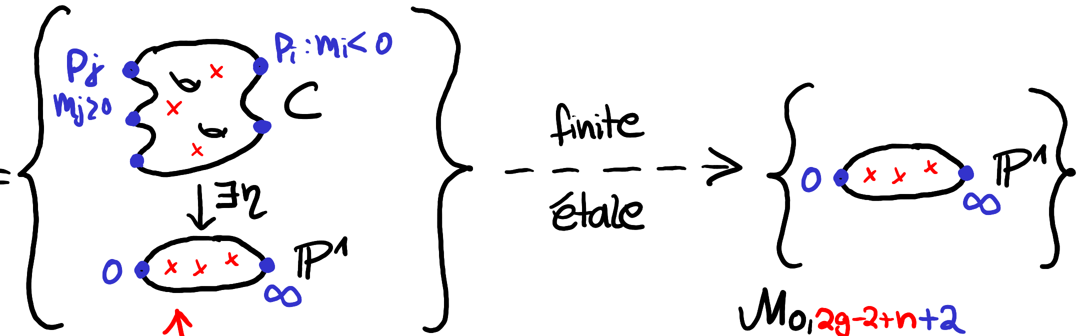
<u>Trivial case</u> $\mu = (0,0,\dots,0)$

$$\Rightarrow \mathcal{H}_g^{k=0}(\underline{0}) = \mathcal{M}_{g,n}$$

Codimension 0

<u>General case</u> $\mu = (m_1,\dots,m_n) \neq 0$

$$\rightsquigarrow \mathcal{H}_g^{k=0}(\mu) =$$



Riemann-Hurwitz:
branch degree away from $0,\infty$
$= 2g-2+n$

$$\Rightarrow \dim \mathcal{H}_g^{k=0}(\mu) = \dim \mathcal{M}_{0,2g-2+n+2} = 2g+n-3$$

$$\cap$$

$$\dim \mathcal{M}_{g,n} = 3g-3+n$$

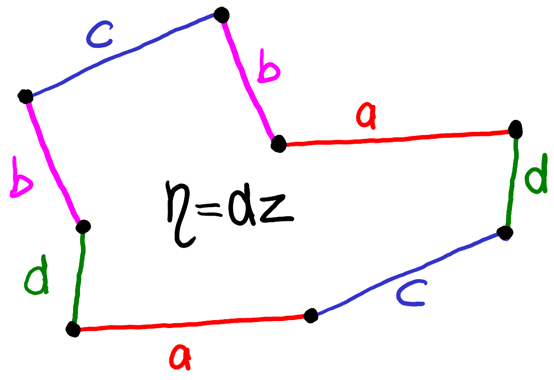
$$\Rightarrow \operatorname{codim} \mathcal{H}_g^{k}(\mu) = g$$

Codimension g

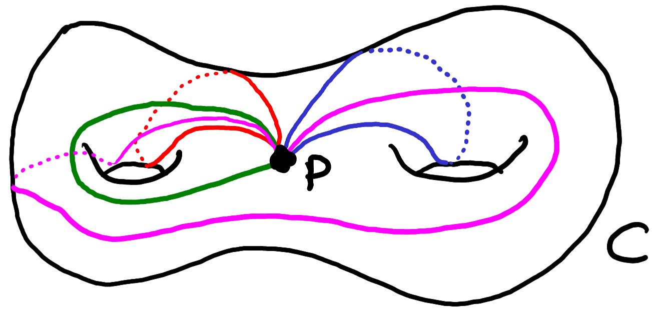
§2.2 Strata of differentials $(k=1)$

§2.2.1. Holomorphic case

Exa $g=2, \mu=(2) \rightsquigarrow (C,p)$ with $\eta \in \Gamma(\omega_C)$ st. $\mathrm{div}(\eta)=2\cdot p$



translation surface



Curve with saddle connections

$$\Rightarrow \dim \mathcal{H}_2^{k=1}(2) = \underset{abcd}{4} - \underset{\mathbb{C}^*}{1} = 3 = 2g-2+n$$

In general: $\mu=(m_1,\dots,m_n)$ w/ $m_i \geq 0$

$$\Rightarrow \operatorname{codim} \mathcal{H}_g^{k=1}(\mu) = g-1$$

Codimension $g-1$

§2.2.2 Meromorphic case

$\mu = (m_1, \dots, m_n)$ s.t. $\exists i$ with $m_i < 0$

$\Rightarrow \operatorname{codim} \mathcal{H}_g^{k=1}(\mu) = g$

Proof Deformation theory [Polishchuk, Farkas–Pandharipande]

Codimension g again

Special case:

$\mu = (-1, m_2, \dots, m_n)$, $m_i \geq 0$

$\Rightarrow \mathcal{H}_g^{k=1}(\mu) = \emptyset$

Resp. $\eta = 0$ by residue theorem

$\lightning$ to pole of order 1.

empty moduli space

[**Thm** [Kontsevich–Zorich]
Classification of connected components of $\mathcal{H}_g^{k=1}(\mu)$]

§2.3. Strata of higher differentials ($k > 1$)

Thm [S., Bainbridge–Chen–Gendron–Grushevsky–Möller]
For $k \geq 1$, all components of $\mathcal{H}_g^k(\mu)$ have codimension g, except those parameterizing k-th powers of holomorphic 1-differentials which have codimension $g-1$.

$$\mathcal{H}_2^{k=2}(2,2) = \underbrace{\mathcal{H}_2^{k=1}(1,1)}_{\text{codim } g-1 = 1} \sqcup \underbrace{Q_2(2,2)}_{\text{codim } g = 2}$$

Connected components

- Known for $k=2$ [Lanneau, Boissy]
- Open with partial results [Chen-Gendron] for $k>2$.

$k>2$: unknown territory

Summary

- Strata of k-differentials come in different shapes and sizes, with quite different modular interpretations

codim 0

codim $g-1$

codim g

- $\dim \mathcal{H}_g^k(\mu)$ known, but even $\pi_0(\mathcal{H}_g^k(\mu))$ unknown for k arb.

<u>This talk:</u> $\chi^{orb}(\mathcal{H}_g^k(\mu))$

§3. A conjecture on the Euler characteristic of $\mathcal{H}_g^k(\mu)$

§3.1 Quick primer on orbifold Euler characteristics

X complex variety $\Rightarrow \chi^{top}(X) = \sum_{i \geq 0} (-1)^i \cdot \dim H^i(X,\mathbb{Q}) \in \mathbb{Z}$

$\Big\}$ generalization

$\mathcal{X}$ orbifold / DM-stack $\Rightarrow \chi^{orb}(\mathcal{X}) \in \mathbb{Q}$

Univ. properties (a) $\mathcal{X} = \coprod_j \mathcal{X}_j \quad \Rightarrow \chi^{orb}(\mathcal{X}) = \sum_j \chi^{orb}(\mathcal{X}_j)$

loc. closed.

(b) $\mathcal{X} \longrightarrow \mathcal{Y}$ smooth fibration w/ fiber $\mathcal{F}$

$\Rightarrow \chi^{orb}(\mathcal{X}) = \chi^{orb}(\mathcal{Y}) \cdot \chi^{orb}(\mathcal{F})$

(c) $\chi^{orb}(X) = \chi^{top}(X)$ for X variety.

Exa $pt \to BG = [pt/G]$ principal G-bundle $\quad$ (G fin. gp.)

$$\Rightarrow \chi^{orb}(BG) \overset{(b)}{=} \chi^{orb}(pt) / \chi^{orb}(G) \overset{(c)}{=} 1/|G|$$

§3.2. Polynomiality of $\chi^{orb}(\mathcal{H}_g^k(\mu))$

Conjecture For each $g \geq 0$ there exists a symmetric polynomial

$$E_g \in \mathbb{Q}[x_1, x_2, x_3, \ldots]$$

of degree $2g$ in formal variables $(x_i)_{i \geq 1}$ such that

$$\chi^{orb}(\mathcal{H}_g^k(\mu)) = (-1)^n \cdot \frac{(2g-3+n)!}{(2g-3)!} \cdot E_g(\mu),$$

where as before $n = \mathrm{len}(\mu)$.

Small print
For $m = 2g-3 < 0$, define $m! = 1$ in the denominator

Examples

The symmetric polynomial E_g can be expressed in terms of

$$e_1 = \sum_i x_i \ , \ e_2 = \sum_{i<j} x_i x_j \ , \ e_3 = \sum_{i<j<m} x_i x_j x_m , \ldots$$

elementary symmetric polynomials

Assuming the Conjecture, here are the first few values:

$$
\begin{aligned}
E_0 &= -1 \\
E_1 &= \frac{1}{12} + \frac{1}{12} e_2 \\
E_2 &= -\frac{1}{240} + \frac{1}{192} e_1^2 - \frac{1}{3840} e_1^4 - \frac{1}{144} e_2 + \frac{1}{1440} e_2 e_1^2 - \frac{1}{360} e_2^2 + \frac{1}{288} e_3 e_1 - \frac{1}{720} e_4 \\
E_3 &= \frac{1}{1008} - \frac{1}{768} e_1^2 + \frac{19}{73728} e_1^4 - \frac{19}{4128768} e_1^6 \\
&\quad + \frac{1}{480} e_2 - \frac{19}{23040} e_2 e_1^2 + \frac{19}{860160} e_2 e_1^4 + \frac{1}{720} e_2^2 - \frac{1}{7680} e_2^2 e_1^2 \\
&\quad + \frac{1}{3360} e_2^3 - \frac{7}{5760} e_3 e_1 + \frac{17}{107520} e_3 e_1^3 - \frac{23}{40320} e_3 e_2 e_1 - \frac{1}{10080} e_3^2 \\
&\quad + \frac{1}{1440} e_4 + \frac{11}{80640} e_4 e_1^2 + \frac{1}{2016} e_4 e_2 - \frac{1}{2240} e_5 e_1 + \frac{1}{5040} e_6
\end{aligned}
$$

Exa

$$\chi^{orb}(\mathcal{H}_2^{k=1}(3,-1)) = (-1)^2 \cdot \frac{(2\cdot 2-3+2)!}{(2\cdot 2-3)!} \cdot E_2(\mu)$$

$\emptyset$ //

$$= 3! \cdot \left(-\frac{1}{240} + \frac{1}{192}\cdot(3-1)^2 - \frac{1}{3840}\cdot(3-1)^4 - \frac{1}{144}\cdot(3\cdot(-1)) + \frac{1}{1440}\cdot(3\cdot(-1))\cdot(3-1)^2 - \frac{1}{360}\cdot(3\cdot(-1))^2\right)$$

$$= 0$$

Basic observations

- $\chi^{orb}(\mathcal{M}_g) = E_g(0) = \frac{B_{2g}}{2g\cdot(2g-2)}$ [Harer–Zagier]

  $= \mathcal{H}_g^{k=0}, n=0$

- $\mathcal{H}_g^{-k}(-\mu) = \mathcal{H}_g^{k}(\mu) \Rightarrow E_g$ even polyn.

  $E_g(-\underline{x}) = E_g(\underline{x})$

§4. Evidence & calculations

§4.1. General results

Prop The conjecture is true for $g=0$ and $g=1$.

Pf $\boxed{g=0}$ $\mathcal{H}_g^k(\mu) = \mathcal{M}_{0,n}$ $\leftarrow$ $(\mathbb{P}^1, p_1,\dots,p_n) \in \mathcal{M}_{0,n} \Rightarrow \eta = \prod_{i=1}^{n}(z-p_i)^{m_i} dz^{\otimes k}$ exists

$\mathcal{M}_{0,n} \longrightarrow \mathcal{M}_{0,n-1}$ fibration w/ fiber $\underbrace{\mathbb{P}^1 \setminus \{n-1 \text{ pts}\}}_{\chi = 2-(n-1) = -(n-3)}$

$$\Rightarrow \chi(\mathcal{M}_{0,n}) = (-1)\cdot(n-3)\cdot \chi(\mathcal{M}_{0,n-1})$$

$$\cdots = (-1)^{n-3}\cdot (n-3)! \quad \checkmark$$

$\boxed{g=1}$ $\mu = (0,\dots,0) \rightsquigarrow$ follows from Harer-Zagier

If e.g. $m_1 \neq 0$, consider the forgetful map

$$\pi : \mathcal{H}_1^k(\mu) \longrightarrow \underset{\uparrow \chi^{orb} = -1/12}{\mathcal{M}_{1,1}}, \quad (C, p_1,\dots,p_n) \mapsto (C, p_1)$$

$$\pi^{-1}(E,p) = \left\{ (p_2,\dots,p_n) \in E^{n-1} \;\middle|\; \begin{array}{l} p_2^{\oplus m_2} \oplus \cdots \oplus p_n^{\oplus m_n} = 0 \text{ in } (E,p) \\ \boxed{p_i \neq p_j \;\forall i,j, \quad p_i \neq p \;\forall i} \; (\ast) \end{array} \right\}$$

$n>2$, omit condition $(\ast) \rightsquigarrow \chi = 0$

$\rightsquigarrow$ inclusion-exclusion argument for collisions of $p, p_2,\dots,p_n$ $\Rightarrow$ formula for χ^{orb} from conjecture $\square$

[Independent work in progress by Battistella-Nabijou].

$\rightsquigarrow$ What about $g \geq 2$?

§4.2 Computational verification

General ansatz

$$E_g = \sum_{\substack{m=0,2,4,\dots,2g \\ \lambda \vdash m}} c_\lambda \cdot \prod_{\lambda_i \in \lambda} e_{\lambda_i}$$

↑ unknown rat'l number

g	0	1	2	3	4	5	6
$\#c_\lambda$	1	3	8	19	41	83	160

number of free parameters c_λ

Known value of $\chi^{orb}(\mathcal{H}_g^k(\mu))$

⇝

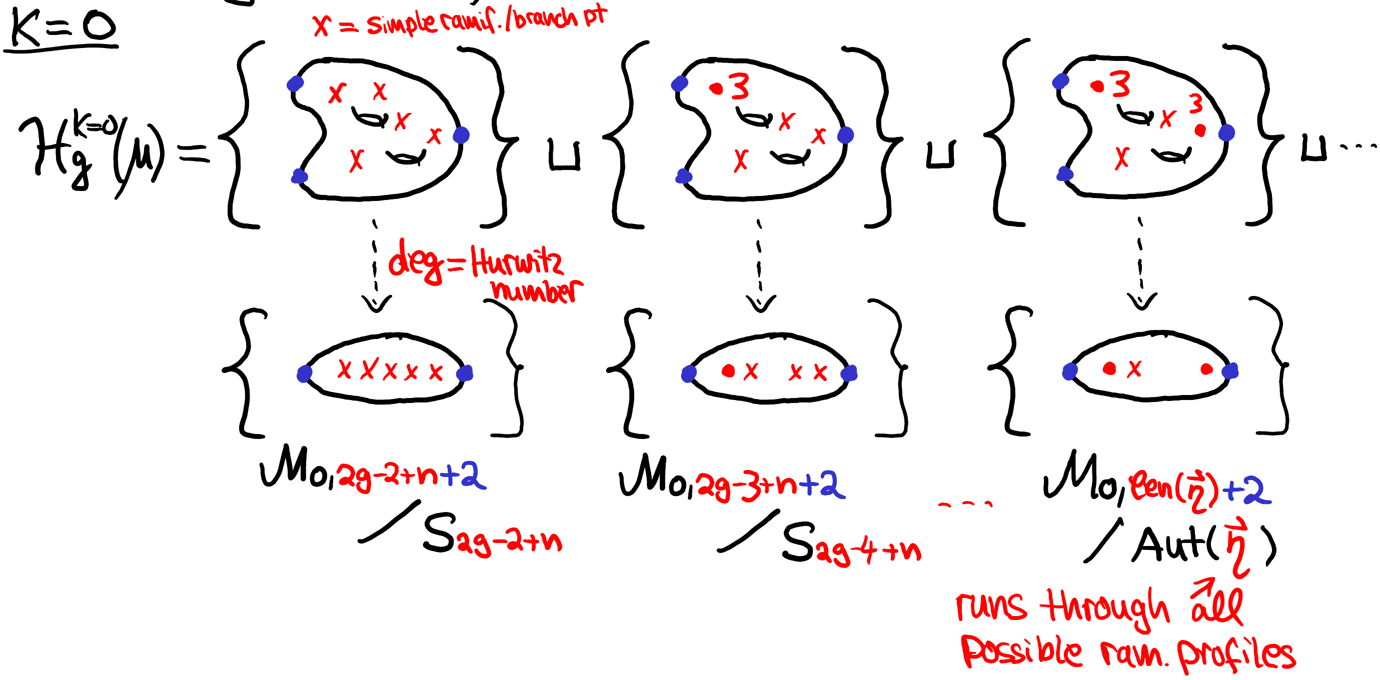
affine lin. equation for c_λ

new ⇝ help to determine formula of E_g

old ⇝ help to verify Conjecture

Calculating $\chi^{orb}(\mathcal{H}_g^k(\mu))$ in examples

$\underline{k=0}$



$$\Rightarrow \chi^{orb}(\mathcal{H}_g^{k=0}(\mu)) = \sum_{\vec{\eta}} \frac{1}{|\mathrm{Aut}(\vec{\eta})|} \cdot \chi(\mathcal{M}_{0,\mathrm{len}(\vec{\eta})+2}) \cdot \mathrm{HurNum}(\mu^+, \mu^-, \vec{\eta})$$

↗ Calculate e.g. using representat. theory of symmetric gps. [Zekun Ji – Master thesis]

$\underline{K=1}$ For free: $\chi^{orb}(\mathcal{H}_g^{k=1}(-1, a_2, \ldots, a_n)) = 0$ for $\underline{a} \vdash 2g-1$

More general:

Thm [Costantini–Möller–Zachhuber]
Algorithm for calculating $\chi^{orb}(\mathcal{H}_g^{k=1}(\mu))$.
Geometric input: intersection numbers on $\overline{\mathcal{M}}_{g',n'}$

Idea $\overline{\mathcal{M}}$ compact, cplx dim. d

$$\Rightarrow \chi^{orb}(\overline{\mathcal{M}}) = (-1)^d \cdot \int_{\overline{\mathcal{M}}} c_d(\Omega^1_{\overline{\mathcal{M}}})$$

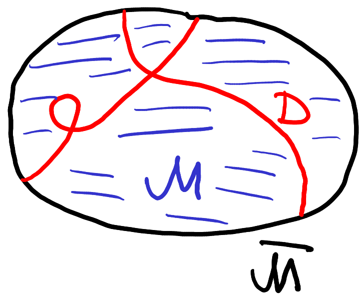
top Chern class

cotangent bdle

Gauss-Bonnet formula

Variant $\mathcal{M} \subseteq \overline{\mathcal{M}}$ open st. $D = \overline{\mathcal{M}} \setminus \mathcal{M}$ is normal crossing div.

$$\Rightarrow \chi^{orb}(\mathcal{M}) = (-1)^d \cdot \int_{\overline{\mathcal{M}}} c_d(\underbrace{\Omega^1_{\overline{\mathcal{M}}}(\log D)}_{\text{sheaf of differentials with log poles along } D})$$



[BCGGM] construct nc compactification $\overline{\mathcal{M}}$ of $\mathcal{H}_g^{k=1}(\mu)$

multi-scale differentials

[CMZ] compute $c_d(\Omega^1_{\overline{\mathcal{M}}}(\log D))$
& implement intersect. number in Sage Math

diffstrata package

Recently [Costantini–Möller–Schwab] generalize to $k > 1$.

§5. Open directions

- Gather more data $(E_4, E_5, \ldots) \rightarrow$ computations get hard!
- Guess formula for E_g → data on my website
- Proof ??????

Thanks for your attention!