Logarithmic tautological rings

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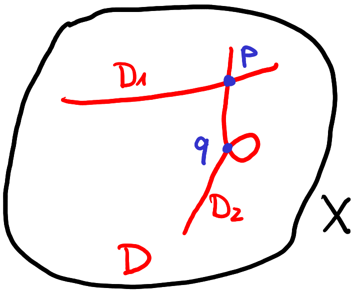
§0 Motivation

(X, D) smooth space with normal crossings divisor (smooth log smooth)



E.g. $(X = \overline{\mathcal{M}}_{g,n}$, $D = \partial \overline{\mathcal{M}}_{g,n})$

mod. space of stable curves

locus of singular curves

$\rightsquigarrow$ stratification $X = \bigsqcup_\sigma S_\sigma$ into loc. closed $S_\sigma \subseteq X$.

$\boxed{Q1}$ How to intersect classes $[\overline{S_\sigma}] \in CH^*(X)$?

$\rightsquigarrow$ nice combinatorial formalism ?

E.g. $[D_1] \cdot [D_2] = [p] \in CH^2(X)$ above.

Next problem sometimes $U = X \setminus D$ is canonical, but it's (partial) compactification X is not !

Exa $U = \mathcal{A}_g$: moduli space of princ. polarized abel. var of dim g

$\rightsquigarrow X = \overline{\mathcal{A}}_g$: different birat'l models $(\overline{\mathcal{A}}_g^{PC}, \overline{\mathcal{A}}_g^{Alexeev}, \ldots)$

In good cases:

$$\begin{tikzcd} & \exists \widehat{X} \arrow[dl, dashed] \arrow[dr, dashed] & \\ X_1 & & X_2 \\ & U \arrow[ul, phantom, "\subseteq"] \arrow[ur, phantom, "\subseteq"] & \end{tikzcd}$$

any two X_1, X_2 receive maps from common space $\widehat{X}$ and $\widehat{X} \to X_i$ is a log blow-up

think: iterated blow-up of smooth strata closures

Def (Holmes–Pixton–S.)

(X, D) smooth nc pair.

$$\rightsquigarrow \log CH^*(X, D) := \varinjlim_{\substack{(\widehat{X}, \widehat{D}) \to (X, D) \\ \text{log blow-up}}} CH^*(\widehat{X})$$

trans. maps = pullb. $\widehat{\pi}^*$

$\widehat{\widehat{X}} \xrightarrow{\widehat{\pi}} \widehat{X} \longrightarrow X$

logarithmic Chow ring

Basic properties

- $\log CH^*(X,D) = \{(\hat X,\alpha) : \hat X \to X \text{ log blow-up}, \alpha \in CH^*(\hat X)\} / (\hat X,\alpha) \sim (\hat{\hat X}, \hat\pi^*\alpha)$ for $\hat{\hat X} \xrightarrow{\hat\pi} \hat X \to X$.

- $\log CH^*(X,D)$ is $\mathbb{Q}$-algebra,

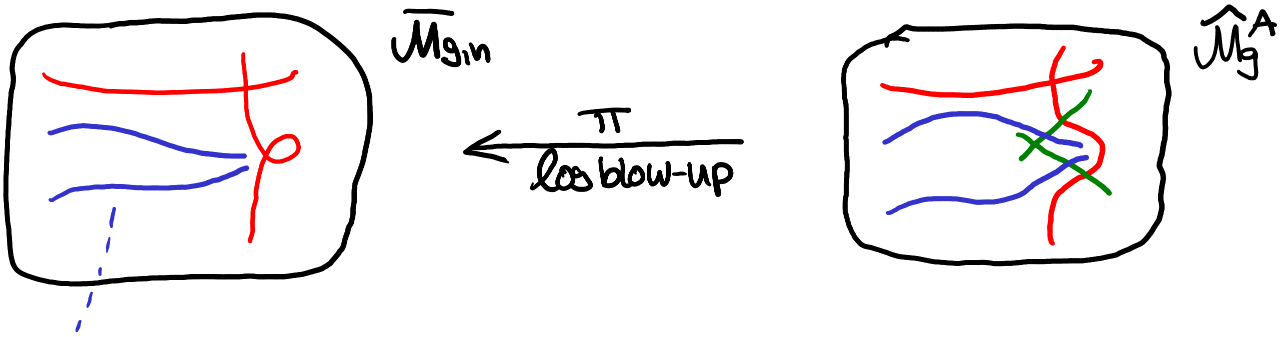
$$\begin{aligned} CH^*(X) &\hookrightarrow \log CH^*(X,D) \\ \alpha &\longmapsto [(X,\alpha)] \end{aligned}$$

- $\log CH^*(X,D) \twoheadrightarrow CH^*(X)$ $\quad$ $\mathbb{Q}$-linear

$$[(\hat X \xrightarrow{\pi} X, \alpha)] \longmapsto \pi_*\alpha.$$

Applications & History

- [HPS] defined the logarithmic double ramification cycle



$DR_g^\circ(A) = \{(C,p_1,\dots,p_n) : \mathcal{O}_C(\sum a_i p_i) \cong \mathcal{O}_C\}$ $\qquad$ $\widehat{DR}_g(A) \in CH^g(\widehat{\mathcal{M}}_g^A)$

[Holmes]

$A=(a_1,\dots,a_n)\in\mathbb{Z}^n$ with $\sum a_i = 0$ $\qquad$ smooth

$$\rightsquigarrow \log DR_g(A) = [(\widehat{\mathcal{M}}_g^A, \widehat{DR}_g(A))] \in \log CH^g(\overline{\mathcal{M}}_{g,n})$$

$$\downarrow \pi_*$$

$$DR_g(A) \in CH^g(\overline{\mathcal{M}}_{g,n}).$$

• [HPS]

$$\log CH^{2g}(\overline{\mathcal{M}}_{g,n})$$
$$\in$$

$$\log DR_g(A) \cdot \log DR_g(B) = \log DR_g(A) \cdot \log DR_g(A+B)$$

False for DR_g

<u>Idea</u> $\mathcal{O}_C(\sum a_i p_i) \cong \mathcal{O}_C$, $\mathcal{O}_C(\sum b_i p_i) \cong \mathcal{O}_C$ $\iff$ $\mathcal{O}_C(\sum a_i p_i) \cong \mathcal{O}_C$, $\mathcal{O}_C(\sum (a_i+b_i) p_i) \cong \mathcal{O}_C$

• [Molcho – Pandharipande – S.]

False for $\mathrm{div}CH^*$

$$DR_g(A) \in \mathrm{div}\log CH^*(\overline{\mathcal{M}}_{g,n})$$

sub-$\mathbb{Q}$-algebra of $\log CH^*$ gen. by $\log CH^1$.

<u>Conjecture</u> [MPS] $\log DR_g(A) \in \mathrm{div}\log CH^*(\overline{\mathcal{M}}_{g,n})$

↳ proven by [Molcho-Ranganathan, Holmes-Schwarz]

• [Cavalieri – Markwig – Ranganathan]

Unknown for DR_g

$$\text{Double Hurwitz Number}_g(A) = \int_{\widehat{\mathcal{M}}_g^A} \log DR_g(A) \cdot br_{g,A}.$$

↳ generalized to include ψ-insertions in [CMS]

• [Holmes – Molcho – Pandharipande – Pixton – S.]
calculate $\log DR_g(A)$ in terms of log-tautological classes on $\overline{\mathcal{M}}_{g,n}$.

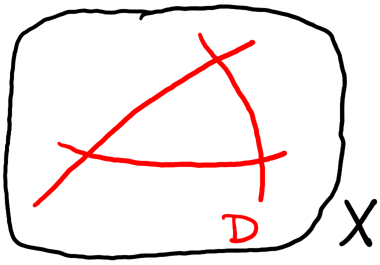
$\boxed{Q2}$ What are log-tautological classes?

⇝ [PRSS]

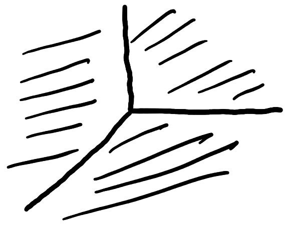
§1 Cone stacks & Artin fans

<u>Case 1</u> X smooth toric variety with torus $T \cong \mathbb{G}_m^n \overset{\text{open}}{\subseteq} X$, $D = X \setminus T$

$\mathbb{G}_m \simeq \mathbb{C}^*$

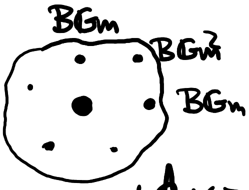


$\rightsquigarrow$



Σ_X fan

$\downarrow q_X$



$\mathcal{A}_X = [X/T]$

<u>Thm</u> (Brion) $\exists$ isom.

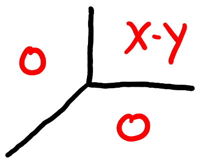
$$\begin{tikzcd} SPP^*(\Sigma_X) \arrow[d, "\Phi", "\simeq"'] \arrow[dr] & \\ CH^*(\mathcal{A}_X) \arrow[r, "q_X^*"] & CH^*(X) \end{tikzcd}$$

Strict piecew. polynomials on Σ_X

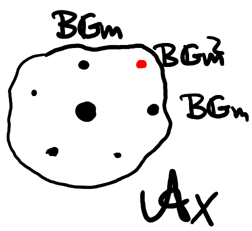
surjective with Kernel $= \langle L(\Sigma) \rangle$

linear fct. on Σ

<u>Exa</u>



$= f \implies \Phi(f) = [B\mathbb{G}_m^2]$



<u>Moreover</u>

log blow-ups of X (or $\mathcal{A}_X$)

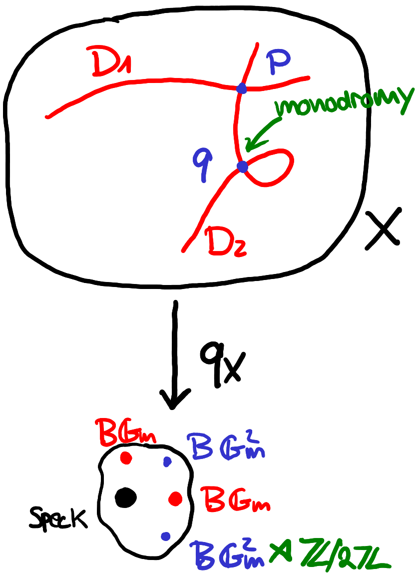
$\widehat{=}$ subdivisions of Σ_X

<u>Case 2</u> (X,D) smooth nc pair

<u>Idea</u> • étale locally around $P \in (X,D)$: D looks toric $\rightsquigarrow \Sigma_P$

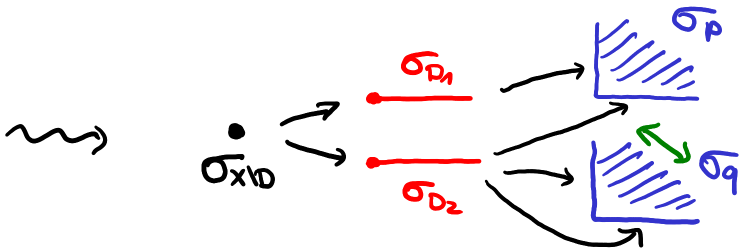
• étale patches glue to (X,D) $\rightsquigarrow$ Σ_P glue to Σ_X

$$\begin{array}{c}(X,D)\\ \downarrow \\ (\mathcal{A}_X,\mathcal{D})\end{array}$$



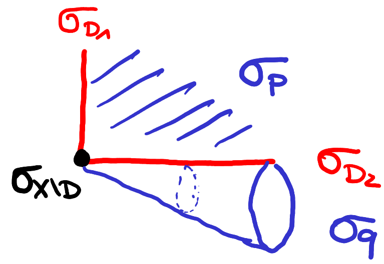
$\mathcal{A}_{(X,D)}$ <u>Artin fan</u>

[Abramovich–Chen–Marcus–Ulirsch–Wise]



$\Sigma_{(X,D)}$: <u>cone stack</u>

[Cavalieri–Chan–Ulirsch–Wise]



<u>Moreover</u>

Subdivisions $\hat{\Sigma} \to \Sigma_{(X,D)}$ $\widehat{=}$ log blow-ups

$$\begin{tikzcd}
(\hat{X},\hat{D}) \arrow[r] \arrow[d, "q_{\hat{X}}"'] \arrow[dr, phantom, "\square"] & (X,D) \arrow[d, "q_X"] \\
(\hat{\mathcal{A}},\hat{\mathcal{D}}) \arrow[r] & (\mathcal{A}_X,\mathcal{D})
\end{tikzcd}$$

__Summary__ $(X,D) \rightsquigarrow \Sigma_{(X,D)}$ cone stack

$$\begin{array}{c}(X,D)\\ \downarrow q_X \\ (\mathcal{A}_X,\mathcal{D})\end{array}\quad \text{Artin fan}$$

__Thm__ [MPS]
$\exists$ isom.

$$sPP^*(\Sigma_{(X,D)}) \xrightarrow[\sim]{\Phi} CH^*(\mathcal{A}_X) \xrightarrow{q_X^*} CH^*(X)$$

$$PP^*(\Sigma_{(X,D)}) \xrightarrow[\sim]{\Phi^{\log}} \log CH^*(\mathcal{A}_X,\mathcal{D}) \xrightarrow{q_X^*} \log CH^*(X,D)$$

functions
$\Sigma_{(X,D)} \to \mathbb{R}$ compatible with
face maps & polynomial
(on all cones / on <u>some</u> subdivision $\hat{\Sigma} \to \Sigma_{(X,D)}$)
sPP* PP*

both q_X^* no longer surjective

Image of $q_X^* \circ \Phi^{(\log)}$: normally decorated strata classes

classes of strata closures in X (or $\tilde{X}$)
decorated by Chern classes of
normal bundles.

$\rightsquigarrow$ $\boxed{Q1}$

__§2__ Applications to moduli spaces of curves

__Def__ The <u>small log-tautological ring</u> $\log R^*_{sm}(\overline{\mathcal{M}}_{g,n})$ is the $\mathbb{Q}$-subalgebra of $\log CH^*(\overline{\mathcal{M}}_{g,n})$ gen. by

- $R^*(\overline{\mathcal{M}}_{g,n}) \subseteq CH^*(\overline{\mathcal{M}}_{g,n}) \subseteq \log CH^*(\overline{\mathcal{M}}_{g,n})$ taut. classes
- $\mathrm{im}(\Phi^{\log} : PP^*(\mathcal{M}_{g,n}^{trop}) \longrightarrow \log CH^*(\overline{\mathcal{M}}_{g,n}))$

$= \Sigma_{(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})}$ moduli space of tropical curves

Thm (HMPPS)

$$\log DR_g(A) = \Big[\exp\big(\eta + \Phi^{\log}(f_L)\big)\cdot \Phi^{\log}(f_P)\Big]_g \in \log R^g_{sm}(\overline{M}_{g,n})$$

$\eta = \sum \frac{a_i^2}{2}\psi_i \in R^1(\overline{M}_{g,n})$ $\qquad$ $f_L, f_P \in PP^*(M^{trop}_{g,n})$ $\qquad$ codim g part.

Thm (PRSS)

$$\Phi^{\log} : PP^*(M^{trop}_{0,n}) \twoheadrightarrow \log CH^*(\overline{M}_{0,n})$$

is surjective, kernel $=$ ideal $(WDVV^{PP}_{0,n})$

Piecew. linear fcts. on $M^{trop}_{0,n}$ mapping to WDVV-rel's as norm. decorated strata classes.

Idea of proof

Construction of [Kapranov] $\rightsquigarrow$ $\overline{M}_{0,n}$ = Chow quot. of $G(2,n)$ by $\mathbb{G}_m^n/\mathbb{G}_m$

$\Rightarrow$ $\exists$ smooth q.proj. toric variety $X_{0,n}$ with torus T: $\begin{pmatrix} p_1 & p_2 & \cdots & p_n \\ q_1 & q_2 & \cdots & q_n \end{pmatrix} \curvearrowleft GL_2$

$$\begin{tikzcd} \overline{M}_{0,n} \arrow[r, hook, "i"] & X_{0,n} \arrow[r] & {[X_{0,n}/T]} = \mathcal{A}_{X_{0,n}} \\ G(2,n) \arrow[u, dashed, "/(\mathbb{G}_m^n/\mathbb{G}_m)"] \arrow[r, hook, "\text{Plücker}"] & \mathbb{P}^{\binom{n}{2}-1} \arrow[u, dashed] & \mathcal{A}_{\overline{M}_{0,n}} \arrow[u, phantom, "\cong"] \end{tikzcd}$$

$\rightsquigarrow$ Artin fans of $\overline{M}_{0,n}$ & $X_{0,n}$ coincide

$\Rightarrow$ $\{$log blow-ups $\widehat{M} \to \overline{M}_{0,n}\} \;\hat{=}\; \{$subdiv of $M^{trop}_{0,n} = \Sigma_{X_{0,n}}\} \;\hat{=}\; \{$log blow-ups $\widehat{X} \to X_{0,n}\}$

Check i^* induces isom. of CH^* on all strata

$\xRightarrow[\text{exact sequence}]{\text{Fulton's blow-up}}$ remains true for $\hat{i} : \widehat{M} \to \widehat{X}$.

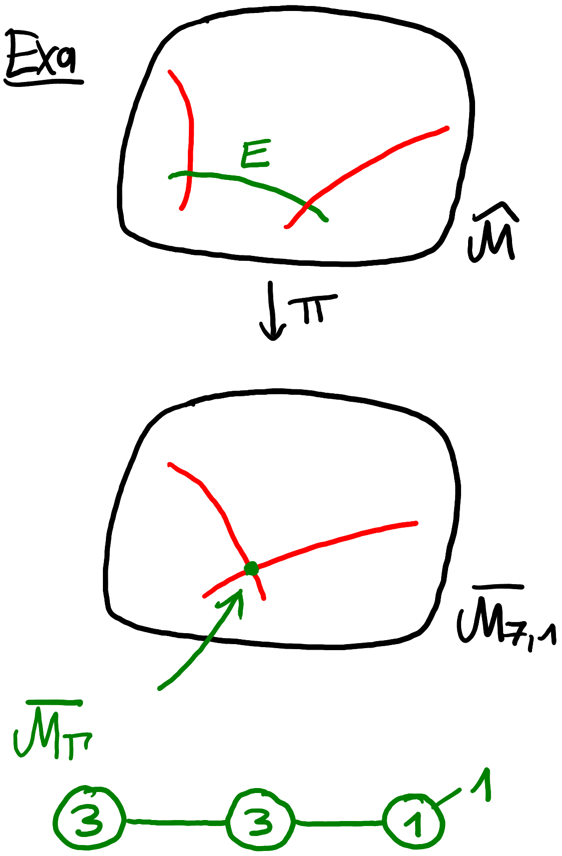
$$\Rightarrow \log CH^*(\overline{M}_{0,n}) = \log CH^*(X_{0,n}) \overset{\text{Brion}}{=} PP^*(\underbrace{\Sigma_{X_{0,n}}}_{= M^{trop}_{0,n}}) \Big/ \big(\underbrace{L(\Sigma_{0,n})}_{\overset{\text{Check}}{=} WDVV^{PP}_{0,n}}\big) \qquad \square$$

§3 Larger log-tautological rings

Problem

some natural classes are missing from $\log R^*_{sm}(\overline{\mathcal{M}}_{g,n})$

Exa



$$\rightsquigarrow \quad \begin{tikzcd} E \arrow[r, hook, "i"] \arrow[d, "\pi_E"'] & \widehat{\mathcal{M}} \\ \overline{\mathcal{M}}_{\Gamma} \end{tikzcd} \quad \alpha = \kappa_1 \otimes 1 \otimes 1$$

$\rightsquigarrow [\widehat{\mathcal{M}}, i_* \pi_E^* \alpha] \in \log CH^2(\overline{\mathcal{M}}_{7,1})$

looks tautological,

but <u>not</u> in $\log R^*_{sm}(\overline{\mathcal{M}}_{7,1})$.

Idea (X,D) snc (for simplicity)

Let $\hat{X} \xrightarrow{\pi} X$ corr. to $\hat{\Sigma} \to \Sigma_{(X,D)}$

$\hat{\sigma} \in \hat{\Sigma} \mapsto \sigma \in \Sigma$

$$\begin{tikzcd} \overline{S}_{\hat{\sigma}} \arrow[r, hook, "i_{\hat{\sigma}}"] \arrow[d, "p_{\hat{\sigma}}"] & \hat{X} \arrow[d, "\pi"] \\ S_{\sigma} \arrow[r, hook] & X \end{tikzcd}$$

strata closures

$$\begin{tikzcd} & \overline{S}_{\hat{\sigma}} \arrow[dl, "p_{\hat{\sigma}}"'] \arrow[r, hook, "i_{\hat{\sigma}}"] \arrow[d, "q_{\hat{\sigma}}"] \arrow[dr, phantom, "\square"] & \hat{X} \arrow[d, "q_{\hat{X}}"] \\ \overline{S}_{\sigma} & \mathcal{B}_{\hat{\sigma}} \arrow[r, hook, "\text{closed}"] & \mathcal{A}_{\hat{X}} \end{tikzcd}$$

Thm (PRSS)

$\exists$ isom.

$$\Psi : \{ f \in sPP^*(\hat{\Sigma}) : f \equiv 0 \text{ outside } \mathrm{Star}_{\hat{\sigma}} \} \xrightarrow{\cong} CH_*(\mathcal{B}_{\hat{\sigma}})$$

Def $\log R^*(X,D)$ is $\mathbb{Q}$-vector space gen. by

$$[\hat\sigma, f, \alpha] := [(\hat X, (i_{\hat\sigma})_*(p_{\hat\sigma}^*\alpha \cap q_{\hat\sigma}^* \Psi(f))]$$

$\hat\sigma$ cone in some $\hat\Sigma \to \Sigma$

f as above

$\alpha \in CH^*(\overline{S}_\sigma)$ decoration $\boxed{\text{CHOICE}}$

$\in \log CH^*(X,D)$

Then

- allowing α = poly in κ, ψ-classes in $CH^*(\overline{\mathcal{M}}_\Gamma)$
  $\rightsquigarrow$ class from above is in $\log R^*(\overline{\mathcal{M}}_{g,n})$

- $\log R^*(X,D)$ completely determined by $(CH^*(\overline{S}_\sigma))_{\sigma \in \Sigma(X,D)}$ & maps between them.

- $\alpha \in CH^*(\overline{S}_\sigma)$ allowed to be arb. elem. of $CH^*(\overline{S}_\sigma)$

  $\rightsquigarrow \log R^*_{CH}(X,D) = \log CH^*(X,D)$

  $\rightsquigarrow [\hat\sigma, f, \alpha]$ give additive generating set.

Thank you for your attention!