Log Chow Theory.

Use log language throughout

Motivation: Given $X \xrightarrow{f} Y$ any morphism we can construct functorial Gysin maps $f^!_E : A_\bullet(Y) \to A_{\bullet-k}(X)$ by embedding the intrinsic normal cone $\mathfrak{C}_f$ into E a vector bundle stack. This gives refined maps for any diagram

$$\begin{tikzcd} W \arrow[r, "f|_V"] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad f^!_{E|_V} : A_\bullet(V) \to A_{\bullet-k}(W).$$

ie a bivariant class $f^!_E \in A_k(X \xrightarrow{f} Y)$.

If instead $f : X \to Y$ is a log morphism all this breaks, log fibre products don't agree with scheme fibre products etc.

One way to make this work (done with restrictions by Leo Herr) would be to pull classes back from $\mathcal{L}og_Y$, Olsson's log stack using the log normal cone $\mathfrak{C}^\ell_{X/Y} := \mathfrak{C}_{X/\mathcal{L}og_Y}$.

Here's a way to generalise this:

$$X \to \mathcal{L}og_Y \xrightarrow[\text{log ét}]{} Y$$

Thm (Fulton-Macpherson): Let $g : Y \to Z$ be smooth relative dimension d and $f : X \to Y$ any map. Then

$$\cdot[g] : A^\bullet(X \xrightarrow{f} Y) \xrightarrow{\sim} A^{\bullet-d}(X \xrightarrow{gf} Z)$$

So if the above theorem held for log smooth morphisms we could construct using the strict map $X \to \mathcal{L}og_Y$ and "pull back" to $X \to Y$.

The theorem doesn't hold for log smooth maps.

But it hints that we at least need to produce groups invariant under log blow ups, (ie use something like b-Chow).

§: Log Geometry

We've seen some of these definitions before in other talks, so I'll try to be brusque.

A log structure on X is:

$$\underset{\substack{\text{sheaf of monoids} \\ \text{in ét topology}}}{M_X} \xrightarrow[\text{morphism of sheaves of monoids}]{\alpha_X} \underset{\substack{\text{multiplicative} \\ \text{monoid str.}}}{\mathcal{O}_X}$$

Such that $\alpha_X^{-1}\mathcal{O}_X^* \to \mathcal{O}_X^*$ is an isomorphism.
We have $\overline{M}_X = M_X / \mathcal{O}_X^*$.

fs $\Rightarrow$ étale locally we have:

$$\begin{tikzcd} U \arrow[r, "ch_U"] \arrow[d, "\text{ét}"'] & \operatorname{Spec} k[P], M_P \\ X & \end{tikzcd} \leftarrow \text{toric log str}$$

such that ch_U identifies $(M_X \xrightarrow{\alpha_X} \mathcal{O}_X)|_U$ with $ch_U^*(M_P \xrightarrow{\alpha_P} \mathcal{O}_{\operatorname{Spec} k[P]})$

F. Kato defines log blowups and log modifications

Blow ups $\tilde{X} \xrightarrow{\pi} X$ fit into diagrams locally:

$$\begin{tikzcd} \tilde{X} \arrow[r, "\pi"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ Bl_K \operatorname{Spec} k[P] \arrow[r, "\pi_0"] & \operatorname{Spec} k[P] \end{tikzcd} \qquad K \trianglelefteq P \text{ a monoid ideal}.$$

eg

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r] \arrow[d] & \mathbb{N}^2 \arrow[d] \\ Bl_0 \mathbb{A}^2 \arrow[r] & \mathbb{A}^2 \end{tikzcd} \qquad \mathbb{P}^1 \text{ has non-toric log str.}$$

Here π_p has a degree zero Gysin map. In general if $\overline{M}_{X,x}$ and $\overline{M}_{\tilde{X},\tilde{x}}$ are isomorphic to $\mathbb{N}^{k_x}$ and $\mathbb{N}^{k_{\tilde{x}}}$ for all x and $\tilde{x}$ there are local Gysin maps gluing to a global degree zero map (have to apply results of Chung-Li to lift from ét top). We say that X and $\tilde{X}$ (or instead just π_X) is locally free.

Dfn: $A^*_\bullet(X) = \varinjlim A_\bullet(\tilde{X})$ over all $\tilde{X} \to X$ a log blowup with $\tilde{X}$ locally free.

§ Push and Pull.

Example that some care is needed:

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "\pi^*"] \arrow[d, "\pi^!"'] & \mathbb{P}^1 \arrow[d, "\pi^!"] \\ \mathbb{P}^1 \arrow[r, "\pi^*"'] & \bullet_{\mathbb{N}^2} \end{tikzcd}$$

does not commute for dimension reasons. The key notions needed are having compatible fundamental class and integrality.

Dfn: Compatible fundamental class: X has compatible fundamental class if X is locally free and for all locally free maps $\tilde{X} \xrightarrow{\pi_X} X$ there is an equality $\pi_X^![X] = [\tilde{X}]$

* Lemma not stated in the talk: For any locally free X

* there is a log blowup $\pi_X : \tilde{X} \to X$ and $V \subset \tilde{X}$ strict

* such that: $[V] = \pi_X^![X]$ and V has c.f.c. "moving lemma"

For definition of integral see eg Nizioł / Ogus.

Thm: (F. Kato) if $f : X \to Y$ is a log morphism then there is a diagram:

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "\pi_X"] & \tilde{Y} \arrow[d, "\pi_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

π_Y a log blowup
π_X a log modification
$\tilde{f}$ integral

So we can define push and pull maps by looking at diagrams:

$$\begin{tikzcd} \overline{X} \arrow[r] \arrow[dr] & \tilde{X} \arrow[r] \arrow[d] & \tilde{Y} \arrow[d] \\ & X \arrow[r] & Y \end{tikzcd}$$

- integral
- log blow ups

and use Gysin maps to move around. You get the usual compatibilities except only saturated pushforward and log flat pull back commute. Nonetheless this is enough to prove the original theorem.

Key compatibility.

Thm 3.2: Given $f : X \xrightarrow{f} Y$ such that Y has c.f.c. and $\pi : \tilde{X} \to X$ a locally free log blowup then $i : \mathfrak{C}^\ell_{\tilde{X}/Y} \hookrightarrow \mathfrak{C}^\ell_{X/Y} \times_X \tilde{X}$ and $i_*([\mathfrak{C}^\ell_{\tilde{X}/Y}]) = \pi^!([\mathfrak{C}^\ell_{X/Y}])$

note that this does not happen naively for dimension reasons:

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r] \arrow[d] & \mathrm{Bl}_0 \mathbb{A}^2 \arrow[d] \\ \mathbb{N}^2 \bullet \arrow[r] & \mathbb{A}^2 \end{tikzcd}$$