Double ramification loci via exact differentials

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Problem

Extend $\mathcal{H}_g^k(\mu)$ to $\overline{\mathcal{M}}_{g,n}$.

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Problem

Describe the closure of $\mathcal{H}_g^k(\mu)$ in $\overline{\mathcal{M}}_{g,n}$ geometrically.

$k = 0$: Admissible covers

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- f is etale away from the nodes and marked points.
- The nodes of C are the preimages of nodes of D.
- If a node identifies x and y then

$$\operatorname{mult}_x f = \operatorname{mult}_y f.$$

The closure of $\mathcal{H}_g^0(\mu)$

Theorem(Harris-Mumford)

A curve (X, p) is contained in the closure of $\mathcal{H}_g^0(\mu)$ if and only if there exists a semistable curve $(\widetilde{X}, p)$ and an admissible cover $f : \widetilde{X} \to C$, $g(C) = 0$ such that

1. $(\widetilde{X}, p)$ is stably equivalent to (X, p).

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Remark

Fix a complete ramification profile σ. The stack of admissible covers of type σ is a compactification of the Hurwitz space $\mathrm{Hur}(\sigma)$. It is not smooth but its normalization always is.

The closure for $k \geq 1$ (BCGGM)

Definition

A level graph $\overline{\Gamma}$ is a stable graph Γ together with a level function $\ell : V(\Gamma) \to \mathbb{Z}_{\leq 0}$.

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A twisted differential (X, η) of type μ compatible with $\overline{\Gamma}$ is a collection of meromorphic differentials $(\eta_v)_{v \in V}$ such that

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4. Global residue condition

The closure for $k \geq 1$ (BCGGM)

Theorem (Bainbridge-Chen-Gendron-Grushevsky-Möller)

A curve (X, p) is in the closure of $\mathcal{H}_g^1(\mu)$ if and only if there exists a twisted differential η of type μ compatible with some level graph $\overline{\Gamma}$ on the dual graph of X.

The closure for $k = 0$ via exact differentials

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- Exact differentials are characterized by the condition

$$\int_\alpha \omega = 0 \text{ for any closed path } \alpha.$$

Hurwitz spaces and exact differentials

Observation

Fix a ramification profile $\sigma = (\sigma_{11}, \ldots, \sigma_{1i_1}, \ldots, \sigma_{d1}, \ldots, \sigma_{di_d})$. Set

$$\mu_{ij} := \begin{cases} \sigma_{ij} - 1, & \text{if i>1} \\ -\sigma_{ij} - 1, & \text{if i=1} \end{cases} .$$

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Idea

Compute limits of rational functions using the BCGGM-compactification $\rightsquigarrow$ need to understand limits of zero periods.

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Remark

The closure of a Hurwitz space inside the BCGGM-compactification yields a smooth compactification.

Example: A degeneration of $\mathcal{H}_3^0(1^4, -4)$

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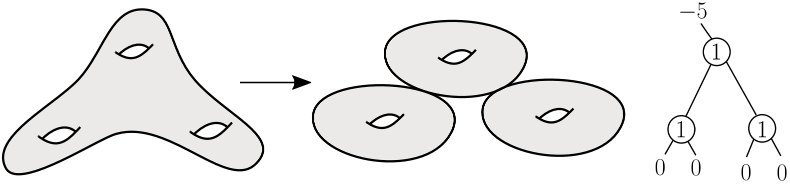

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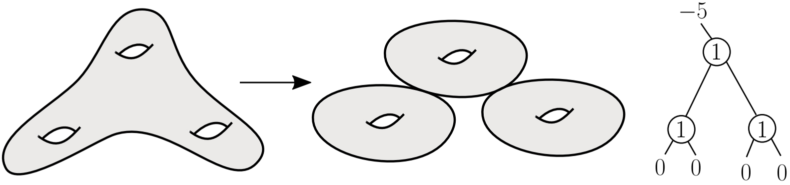

Question

What happens to the condition $\int_\alpha \omega(t) = 0$ in the limit?

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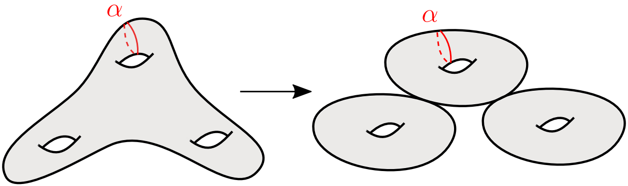

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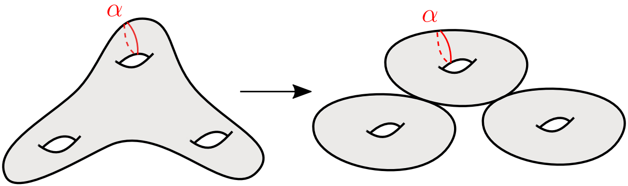

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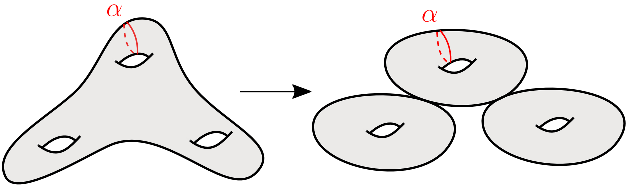

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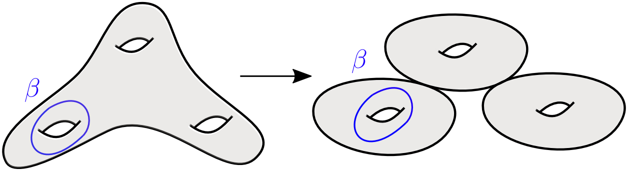

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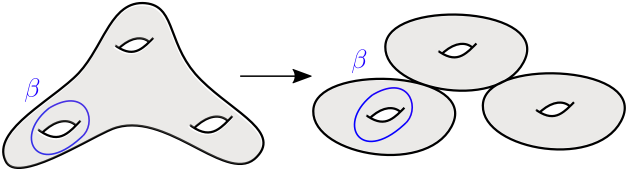

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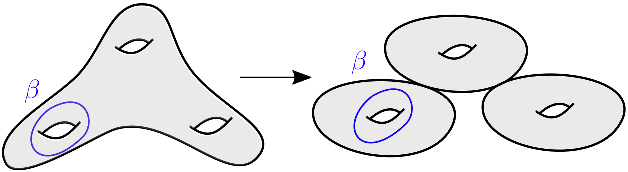

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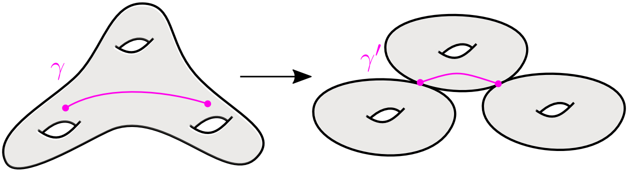

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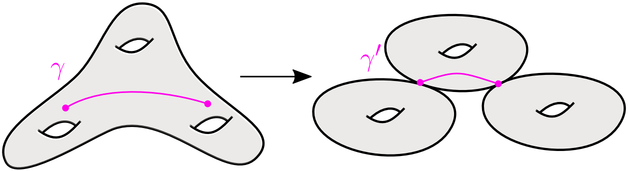

$$\lim_{t \mapsto 0} \int_{\gamma} \omega(t) = \int_{\gamma'} \eta.$$

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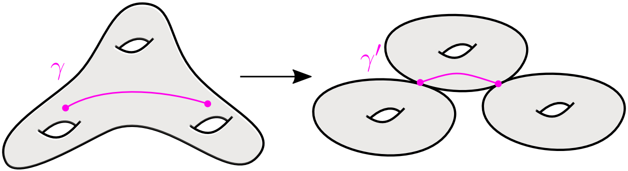

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Consequences

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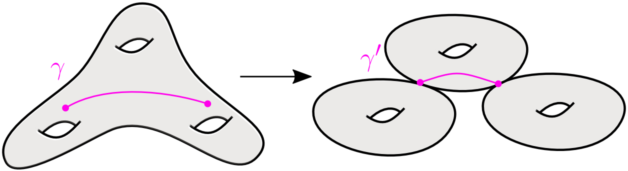


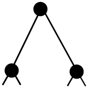
$$\lim_{t \mapsto 0} \int_{\gamma} \omega(t) = \int_{\gamma'} \eta.$$

Consequences

- η is exact on each irreducible component
- For each path (absolute or relative) γ we additionally have $f(\gamma'(1)) - f(\gamma'(0)) = 0$.

Subgraph of level i

Consider $\overline{\Gamma}$ as 1-dimensional cell complex with half-legs for each marked zero of f.

$\overline{\Gamma} =$ 

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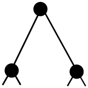
$\overline{\Gamma} =$ 

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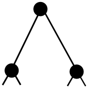
$\overline{\Gamma} =$ 

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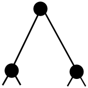
$\overline{\Gamma}_{(0)} =$  , $\overline{\Gamma}_{(\leq -1)} =$ 

The level filtration

Definition

$$L_{(\leq i)}(\overline{\Gamma}) := \mathrm{Im}(H_1(X_{(\leq i)}, z; \mathbb{Z}) \to H_1(X, z; \mathbb{Z}))$$

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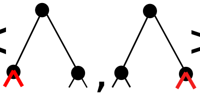
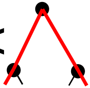
Example:

$L_{(\leq -1)}(\overline{\Gamma}) =$ <  >

$L_{(\leq 0)}(\overline{\Gamma}) = L_{(\leq -1)}(\overline{\Gamma}) \oplus$ <  >

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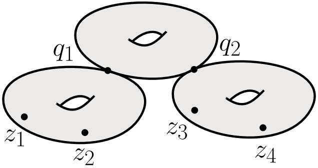
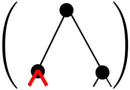
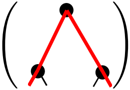


$\mathrm{ev}_f^{(-1)}(\overline{\Gamma})$  $= f(z_1) - f(z_2),$

$\mathrm{ev}_f^{(0)}(\overline{\Gamma})$  $= f(q_1^+) - f(q_2^+)$

Twistable rational functions

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3. If q_2 is a pole of f_{v_2}, then $\ell(v_1) > \ell(v_2)$.
4. If X_v contains a marked zero of f_v, then all zeros of f_v are at marked points or nodes.

The closure via exact differentials

Theorem(- 2020)

A curve (X, p) is contained in the closure of $\mathcal{H}_g(\mu)$ if and only if there exists a level graph $\overline{\Gamma}$ and a twistable rational function of type μ compatible with $\overline{\Gamma}$ on X such that the evaluation morphism vanishes for all levels.

Applications to class computations

- $\pi : \mathcal{C}_g \to \mathcal{M}_g$ universal curve, $p : \mathcal{H}(\mu) \to \mathcal{M}_g$ projection

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Evaluation section

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Evaluation section

$$\begin{aligned} \mathrm{ev} :& \mathcal{R} \otimes \mathcal{O}_{\mathcal{H}(\mu)}(-1) \to \mathcal{O}, \\ & \alpha \otimes \omega \mapsto \int_\alpha \omega \end{aligned}$$

The zero locus consists of exact differentials

$$\{\text{exact differentials}\} = Z(\mathrm{ev})$$

Applications to class computations

- $\pi : \mathcal{C}_g \to \mathcal{M}_g$ universal curve, $p : \mathcal{H}(\mu) \to \mathcal{M}_g$ projection
- $\mathcal{O}_{\mathcal{H}(\mu)}(-1)$ tautological bundle with fiber $\mathbb{C} \cdot \omega$.
- $\mathcal{R} := p^*(\pi_* \underline{\mathbb{Z}} \otimes_{\underline{\mathbb{Z}}} \mathcal{O}_{\mathcal{M}_g})^*$ with fiber $H_1(X; \mathbb{Z})$

Evaluation section

$$\mathrm{ev} : \mathcal{R} \otimes \mathcal{O}_{\mathcal{H}(\mu)}(-1) \to \mathcal{O},$$
$$\alpha \otimes \omega \mapsto \int_\alpha \omega$$

The zero locus consists of exact differentials

$$\{\text{exact differentials}\} = Z(\mathrm{ev})$$

Everything extends to the BCGGM-compactification.