Compactification of Moduli Spaces of Differentials

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- Let $\mu = (m_1, \ldots, m_n)$ be a partition of $2g-2$. Define

$$\Omega\mathcal{M}_g(\mu) = \Big\{(X, \omega) \mid X \text{ is a smooth curve of genus } g,$$

$$\omega \in H^0(X, K), \quad (\omega) = m_1 z_1 + \cdots + m_n z_n\Big\}.$$

- $\Omega\mathcal{M}_g(\mu)$ is called the *moduli space (or stratum) of holomorphic differentials of type* $\mu = (m_1, \ldots, m_n)$.
- $\Omega\mathcal{M}_g = \bigsqcup_{\mu \vdash 2g-2} \Omega\mathcal{M}_g(\mu)$ is the *Hodge bundle* (minus $\{0\}$) on the moduli space $\mathcal{M}_g$ of genus g curves.
- $\Omega\mathcal{M}_g(m_1, \ldots, m_n)$ is an orbifold of dimension $\dim_{\mathbb{C}} \Omega\mathcal{M}_g(m_1, \ldots, m_n) = 2g + n - 1$.

Question

How do we compactify $\Omega\mathcal{M}_g(\mu)$?

- First we should consider differentials up to scale.
- Let $\mathbb{P}\Omega\mathcal{M}_g(\mu)$ be the projectivization of $\Omega\mathcal{M}_g(\mu)$ parameterizing canonical divisors of type μ.

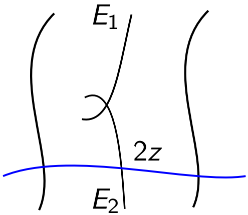
The Hodge bundle compactification

- $\mathbb{P}\Omega\mathcal{M}_g(\mu)$ is a subset of the projectivized Hodge bundle $\mathbb{P}\Omega\mathcal{M}_g$.
- $\mathbb{P}\Omega\mathcal{M}_g$ extends to a projective bundle $\mathbb{P}\Omega\overline{\mathcal{M}}_g$ over $\overline{\mathcal{M}}_g$, parameterizing *stable differentials* which are sections of the dualizing line bundle.
- One can take the closure of $\mathbb{P}\Omega\mathcal{M}_g(\mu)$ in $\mathbb{P}\Omega\overline{\mathcal{M}}_g$ as the *Hodge bundle compactification*.

Lost information: the limit position of zeros.

Example

- Differentials in $\Omega\mathcal{M}_2(2)$ degenerate to two elliptic curves joined at a node:



- The stable differential is vanishing entirely on E_2, so it gives no information for the limit position of z.

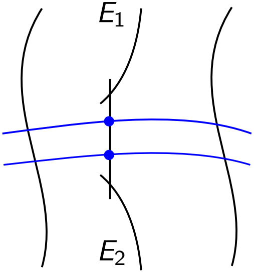
The Deligne-Mumford compactification

- Up to scale, a holomorphic differential ω is determined by its zeros $z_1, \ldots, z_n$.
- Consider $\mathbb{P}\Omega\mathcal{M}_g(\mu)$ as a subset of $\mathcal{M}_{g,n}$, by marking $z_1, \ldots, z_n$.
- One can take the closure of $\mathbb{P}\Omega\mathcal{M}_g(\mu)$ in $\overline{\mathcal{M}}_{g,n}$ as the *Deligne-Mumford compactification*.

Lost information: relative sizes of limit differentials on nonzero components

Example

- Differentials in $\Omega\mathcal{M}_2(1,1)$ degenerate to two elliptic curves joined by a 2-marked $\mathbb{P}^1$:



- As flat tori the sizes of E_1 and E_2 can vary independently, but it is not captured in the Deligne-Mumford compactification.

The incidence variety compactification (IVC)

- The Hodge bundle $\Omega\overline{\mathcal{M}}_{g,n}$ over $\overline{\mathcal{M}}_{g,n}$ parameterizes *pointed stable differentials* $(X, \omega, z_1, \ldots, z_n)$.
- When X is smooth, ω and $z_1, \ldots, z_n$ determine each other (up to scale).
- The closure of $\mathbb{P}\Omega\mathcal{M}_g(\mu)$ in $\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}$ is called the *incidence variety compactification*, denoted by $\mathbb{P}\Omega\overline{\mathcal{M}}^{\text{inc}}_{g,n}(\mu)$.
- Forgetting the marked points, $\mathbb{P}\Omega\overline{\mathcal{M}}^{\text{inc}}_{g,n}(\mu) \to \mathbb{P}\Omega\overline{\mathcal{M}}_g$ determines the Hodge bundle compactification.
- Forgetting the stable differentials, $\mathbb{P}\Omega\overline{\mathcal{M}}^{\text{inc}}_{g,n}(\mu) \to \overline{\mathcal{M}}_{g,n}$ determines the Deligne-Mumford compactification.

Twisted differentials

- The boundary of $\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}^{\mathrm{inc}}(\mu)$ can be described by twisted differentials.
- Let $(X, z_1, \ldots, z_n) \in \overline{\mathcal{M}}_{g,n}$ and $\mu = (m_1, \ldots, m_n)$.
- A *twisted differential of type* μ on $(X, z_1, \ldots, z_n)$ is a collection of (possibly meromorphic) differentials η_v on each irreducible component X_v of X, such that the following conditions hold:

(0) (Vanishing as prescribed) η_v is regular away from the nodes and marked points of X_v. Moreover, if $z_i \in X_v$, then $\operatorname{ord}_{z_i} \eta_v = m_i$.

(1) (Matching orders) If q is a node joining X_1 and X_2, then $\operatorname{ord}_q \eta_1 + \operatorname{ord}_q \eta_2 = -2$.

(2) (Matching residues) If q is a simple polar node joining X_1 and X_2, then $\operatorname{Res}_q \eta_1 + \operatorname{Res}_q \eta_2 = 0$.

Motivation behind twisted differentials

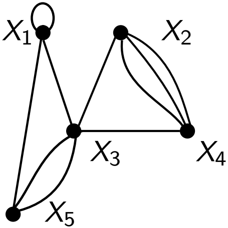
- When differentials of type μ degenerate, the vanishing order along each zero section remains unchanged $\Longrightarrow$ (0) (Vanishing as prescribed).
- $K_X|_{X_i}$ is locally generated by differentials with a simple pole at $q \in X_1 \cap X_2$ for $i = 1, 2$. When twisting by X_i, the zero or pole order of η at q increases by 1 on one branch and decreases by 1 on the other branch $\Longrightarrow$ (1) (Matching orders).
- If there is no twist, a section of K_X with a simple polar node at q has opposite residues on the two branches of $q \Longrightarrow$ (2) (Matching residues).

Level graph

- Let Γ be the *dual graph* of X.
- Order the vertices of Γ, namely, order the irreducible components of X.
- Putting bigger vertices on higher levels, such an order is equivalent to a *level graph* $\overline{\Gamma}$.

Example

A dual graph with three levels $X_1 \sim X_2 > X_3 \sim X_4 > X_5$:



Twisted differentials compatible with a level graph

- Let $\overline{\Gamma}$ be a level graph on $(X, z_1, \ldots, z_n)$.
- A twisted differential η of type μ on X is called *compatible with* $\overline{\Gamma}$, if it satisfies the following conditions:

(3) (Partial order) Suppose a node q joins X_1 and X_2. Then $X_1 > X_2$ iff $\operatorname{ord}_q \eta_1 > \operatorname{ord}_q \eta_2$, i.e., iff $\operatorname{ord}_q \eta_1 \geq 0$. Moreover, $X_1 \sim X_2$ iff $\operatorname{ord}_q \eta_1 = \operatorname{ord}_q \eta_2 = -1$.

Remark

The partial order condition only applies to two irreducible components which intersect.

Motivation behind the partial order condition

- When twisting by X_i, the zero or pole orders at *all* nodes in $X_1 \cap X_2$ increase or decrease simultaneously.
- $X_1 > X_2$ means the twisting coefficient of X_2 is more negative. Namely, nearby differentials degenerate to X_2 faster than to X_1.
- In particular, there cannot exist $X_1 \geq X_2 \geq \cdots \geq X_k \geq X_1$ unless they are all $\sim \Longrightarrow$ (3) (Partial order).

(4) (Global residue condition) For each level L and each connected component Y of the subcurve above L, let $q_1, \ldots, q_b$ be the nodes of Y that connect to the level L. Then

$$\sum_{j=1}^{b} \operatorname{Res}_{q_j} \eta = 0.$$

Pointed stable differentials are shadows of twisted differentials

- The incidence variety compactification $\mathbb{P}\Omega\overline{\mathcal{M}}^{\mathrm{inc}}_{g,n}(\mu) \subset \mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}$ parameterizes pointed stable differentials $(X, \omega, z_1, \ldots, z_n)$.
- Suppose η is a twisted differential of type μ compatible with a level graph $\overline{\Gamma}$ on $(X, z_1, \ldots, z_n)$.
- Define a stable differential ω on X by setting $\omega = \eta$ on top level components and $\omega = 0$ on lower level components.
- The resulting $(X, \omega, z_1, \ldots, z_n)$ is called the *pointed stable differential associated with* η.
- A pointed stable differential can be associated with different twisted differentials by scaling on lower level components.

Theorem (BCGGM1, 2018)

A pointed stable differential $(X, \omega, z_1, \ldots, z_n)$ is contained in $\mathbb{P}\Omega\overline{\mathcal{M}}^{\text{inc}}_{g,n}(\mu)$ iff there exists a level graph $\overline{\Gamma}$ and a compatible twisted differential η of type μ such that $(X, \omega, z_1, \ldots, z_n)$ is associated with η.

Remark

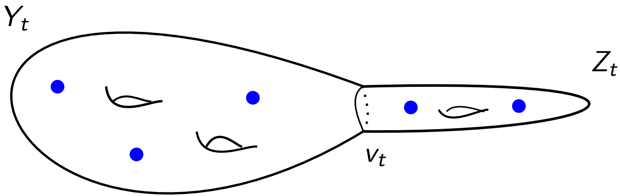
It suggests that we should consider a refined moduli space parameterizing twisted differentials with level graphs.

Remark

[Farkas-Pandharipande, 2018] studied the Deligne-Mumford strata compactification without imposing the global residue condition, which contains extra components in the boundary.

Motivation behind the global residue condition

- Suppose a one-parameter family $\mathcal{X}$ of differentials (X_t, ω_t) degenerate to a nodal curve X at $t = 0$.
- Suppose X has a separating node q joining two components Y and Z.
- Suppose $\lim_{t\to 0} \omega_t|_Y = \eta_Y$ is a holomorphic differential, and $\lim_{t\to 0} (t^{-\ell}\omega_t)|_Z = \eta_Z$ is a meromorphic differential, $\ell \in \mathbb{Z}^+$.
- It means we consider the *twisted* relative dualizing line bundle $\omega_{\mathcal{X}}(-\ell Z)$.

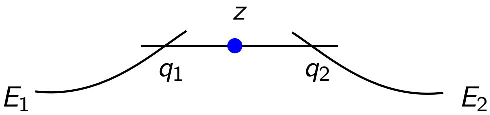


- Let v_t be the vanishing cycle on X_t that shrinks to q.
- q is separating $\Longrightarrow v_t = 0 \in H_1(X_t; \mathbb{Z})$.
- $\int_{v_t} \omega_t = 0 \Longrightarrow \int_{v_t} t^{-\ell}\omega_t = 0 \Longrightarrow \operatorname{Res}_q \eta_Z = 0$ as $t \to 0$.

- In general, suppose $\lim_{t\to 0} t^{-\ell_i}\omega_t|_{X_i} = \eta_i$ on each irreducible component X_i for some $\ell_i \geq 0$.
- $X_i \mapsto -\ell_i$ determines a level graph $\overline{\Gamma}$.
- Use relations of vanishing cycles level by level $\Longrightarrow$ (4) (The global residue condition).

Extra components without the global residue condition

- Consider the stratum $\Omega\mathcal{M}_2(2)$.
- Let (X, z) be two elliptic curves E_1, E_2 joined by a marked $\mathbb{P}^1$ at two nodes q_1, q_2:

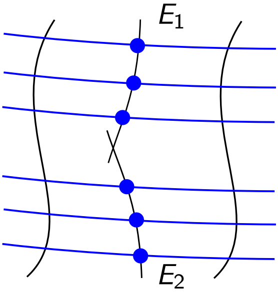


- A twisted differential η on X is holomorphic on E_1, E_2, and $(\eta_{\mathbb{P}^1}) = 2z - 2q_1 - 2q_2$.
- Setting $z = 1, q_1 = 0, q_2 = \infty$,

$$\eta_{\mathbb{P}^1} = c \cdot \frac{(u-1)^2}{u^2} du.$$

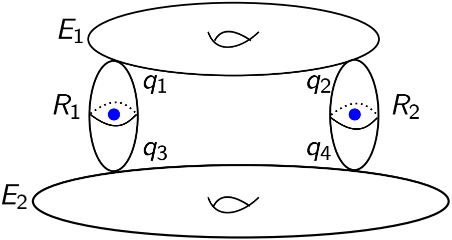
$\Longrightarrow \mathrm{Res}_{q_i}\, \eta_{\mathbb{P}^1} \neq 0 \Longrightarrow \eta$ violates the global residue condition.

- The locus of such (X, z) gives an extra component in the strata compactification by [Farkas-Pandharipande].
- As degeneration of Weierstrass points, the limit of z must be 2-torsion to q_i in E_i, in total 6 such torsion points, equal to the number of Weierstrass points on a curve of genus two:



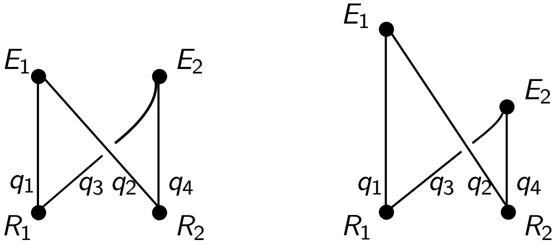
Relative sizes of limit differentials

- Consider the stratum $\Omega\mathcal{M}_3(2,2)$.
- Let (X, z_1, z_2) be the following:



- A twisted differential η on X is holomorphic on E_1, E_2, and $(\eta_{R_1}) = 2z_1 - 2q_1 - 2q_3, \quad (\eta_{R_2}) = 2z_2 - 2q_2 - 2q_4$.
- $E_1, E_2 > R_1, R_2$ and $\mathrm{Res}_{q_j}\, \eta_{R_i} \neq 0$.

- The global residue condition $\Longrightarrow R_1 \sim R_2$ and $\mathrm{Res}_{q_1}\,\eta_{R_1} + \mathrm{Res}_{q_2}\,\eta_{R_2} = 0, \quad \mathrm{Res}_{q_3}\,\eta_{R_1} + \mathrm{Res}_{q_4}\,\eta_{R_2} = 0.$
- Compatible level graphs:



- Differentials in $\Omega\mathcal{M}_3(2,2)$ degenerate to R_1 and R_2 in the same speed.

$\mathbb{P}\Omega\overline{\mathcal{M}}^{\text{inc}}_{g,n}(\mu)$ can have bad singularities

Example

- Let (X, η) be a twisted differential of type μ with a rhombus level graph.
- Suppose the two components X_1 and X_2 in the intermediate level are not bounded by the GRC.
- Rescaling $[\eta_1, \eta_2]$ as $[\lambda_1\eta_1, \lambda_2\eta_2]$ for $[\lambda_1, \lambda_2] \in \mathbb{P}^1$ gives the same associated pointed stable differential in $\mathbb{P}\Omega\overline{\mathcal{M}}^{\text{inc}}_{g,n}(\mu)$.
- In general $\mathbb{P}\Omega\overline{\mathcal{M}}^{\text{inc}}_{g,n}(\mu)$ can have singularities arising from such a small contraction (not $\mathbb{Q}$-factorial).

Example

- The stratum $\mathbb{P}\Omega\mathcal{M}_{3,1}(4)$ consists of two connected components $\mathbb{P}\Omega\mathcal{M}_{3,1}(4)^{\text{hyp}}$ and $\mathbb{P}\Omega\mathcal{M}_{3,1}(4)^{\text{odd}}$, parameterizing respectively hyperelliptic curves with a Weierstrass point and non-hyperelliptic curves with a hyperflex point.
- Consider $(X, \omega, z) \in \mathbb{P}\Omega\overline{\mathcal{M}}_{3,1}^{\text{inc}}(4)$, where X consists of a genus two component X_2 union a genus zero component X_0 at two nodes q_1 and q_2, such that $(\omega|_{X_2}) = q_1 + q_2$ and $\omega|_{X_0} = 0$.
- One checks that $(X, \omega, z) \in \mathbb{P}\Omega\overline{\mathcal{M}}_{3,1}^{\text{inc}}(4)^{\text{hyp}} \cap \mathbb{P}\Omega\overline{\mathcal{M}}_{3,1}^{\text{inc}}(4)^{\text{odd}}$.
- In general $\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}^{\text{inc}}(\mu)$ can self intersect.
- There is a unique twisted differential associated with (X, ω, z). Hence to pull the branches apart, we need to further refine twisted differentials.

Question

How to construct a smooth compactification of $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$?

- Add one more structure to twisted differentials, called *prong-matching*.

- Let $q \in X_1 \cap X_2$ be a node of a twisted differential η such that at the two branches $\operatorname{ord}_q \eta_1 = \kappa - 1$ and $\operatorname{ord}_q \eta_2 = -\kappa - 1$ for $\kappa > 0$.
- There are κ many standard coordinates z_1 (and z_2), called *prongs*, such that locally $\eta_1 = d(z_1^\kappa)$ (and $\eta_2 = d(z_2^{-\kappa})$ omitting residue), which differ by κ-th roots of unity.
- A choice of z_1 and z_2 gives a *local prong-matching* at q, and a *global prong-matching* consists of a local prong-matching at every such node.
- The $\mathbb{C}^*$-action on each level of the level graph (to simultaneously projectivize differentials on the same level) induces an equivalence relation between prong-matchings.

- A *multi-scale differential of type* μ consists of a level graph, a twisted differential of type μ compatible with the level graph, and a global prong-matching (up to equivalence).

Theorem (BCGGM3, 2019)

The moduli space $\mathcal{MS}_\mu$ *of multi-scale differentials of type* μ *is a complex orbifold and a proper Deligne-Mumford stack with normal crossing boundary, containing the stratum* $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ *as an open dense subset.*

Remark

We do not know whether $\mathcal{MS}_\mu$ *is projective.*

- We give a blowup description for $\mathcal{MS}_\mu$.
- Let $(\mathcal{X}, \omega)$ be a family of pointed stable differentials of type μ over a *normal* base B with generic fiber smooth.
- For $p \in B$ and any irreducible component X of the fiber X_p, there exists an *adjusting parameter* $h \in \mathcal{O}_{B,p}/\mathcal{O}^*_{B,p}$ such that $\eta_X = (h^{-1}\omega)|_X$ does not identically vanish on X, has prescribed order at every marking $z_i \in X$, and is regular away from the nodes and markings of X.
- h carries the vanishing part of ω along X, and η_X is the corresponding twisted differential restricted to X.
- The (partial) order between two irreducible components X_1 and X_2 is induced by divisibility of their adjusting parameters h_1 and h_2. Namely, $X_1 \geq X_2$ if $h_1 \mid h_2$.

- For any two adjusting parameters, we want one of them to divide the other, thus extending the partial order to a full order (i.e. a level graph).
- For a small neighborhood U of $p \in B$, define the *disorderly ideal* $\mathcal{D}_U \subset \mathcal{O}_{U,p}$ by $\mathcal{D}_U = \prod (h_i, h_j)$ where the product ranges over all pairs of irreducible components of the fiber X_p.
- Let $\widetilde{U}$ be the blowup of U along $\mathcal{D}_U$. Over $\widetilde{U}$ the ideal $(\widetilde{h}_i, \widetilde{h}_j)$ becomes principal, hence gives the desired divisibility relation.
- These $\widetilde{U}$ can be glued together to form a global *disorderly blowup space* $\widetilde{B}$.
- Over $\widetilde{B}$ there is a corresponding family of multi-scale differentials of type μ.
- There is some flexibility to define $\mathcal{D}_U$, e.g. replacing (h_i, h_j) by arbitrary $(h_{i_1}, \ldots, h_{i_k})$. The resulting disorderly blowup spaces are the same.

- Let $\widetilde{\mathbb{P}\Omega\mathcal{M}}_{g,n}(\mu)$ be the *normalization* of the disorderly blowup of the *normalization* of the IVC $\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}^{\text{inc}}(\mu)$.

Theorem (BCGGM3, 2019)

The moduli space $\mathcal{MS}_\mu$ of multi-scale differentials of type μ can be identified with $\widetilde{\mathbb{P}\Omega\mathcal{M}}_{g,n}(\mu)$.

Remark

In general, adjusting parameters live in étale local rings (not Zariski local rings). Moreover, disorderly ideals locally differ from each other by principal ideals. We have not been able to find a global subscheme to blow up. Hence this blowup description does not guarantee that $\mathcal{MS}_\mu$ is projective.