On connected components of strata of k-differentials

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Basics

Let $k \in \mathbb{N}_{>0}$, $g \in \mathbb{N}$ and $\mu = (m_1, \ldots, m_n)$ a partition of $k(2g-2)$.

The *stratum* $\Omega^k\mathcal{M}_g(m_1, \ldots, m_n)$ is the moduli space of k-differentials (X, ξ) on genus g Riemann surfaces with zeros (and poles) of order m_i.

The stratum are orbifold and each connected component is of dimension either $2g-1+n$ or $2g-2+n$.

Question : ¿What are the connected components of the strata ?

Premilinaries

Given a component of $\Omega^k\mathcal{M}_g(m_1,\ldots,m_n)$ then this gives a component of $\Omega^{k\ell}\mathcal{M}_g(\ell m_1,\ldots,\ell m_n)$ for all $\ell \geq 2$ by taking powers of the k-differentials.

We write $\Omega^k\mathcal{M}_g^{\text{prim}}(m_1,\ldots,m_n)$ the union of components of the strata parametrizing k-differentials which are not (non-trivial) powers.

known results 0

Proposition

All the strata of k-differentials on the Riemann sphere are connected.

Proposition

Let $\Omega^k\mathcal{M}_1(m_1, \ldots, m_n)$ *be a stratum of k-differential. The number of connected components is*

- *the number of common divisors of* m_i *if* $\mu \neq (m, -m)$ *;*
- *the number of strict divisors of* m *if* $\mu = (m, -m)$ *with* $m > 1$*.*

known results 1

Theorem (Kontsevich-Zorich (2003))

The strata of abelian differentials with $\mu > 0$ have at most 3 connected components distinguished by hyperellipticity and parity of a spin structure.

Definition

A component $S \subset \Omega^k\mathcal{M}_g(\mu)$ of a stratum is of hyperelliptic type *if for every $(X, \xi) \in S$ satisfy that X is hyperelliptic and $\iota^*\xi = (-1)^k\xi$.*

Definition

Given an abelian differential ω in $\Omega\mathcal{M}_g(2\ell_1, \ldots, 2\ell_n)$, then the parity of ω is the parity $h^0(X, \mathrm{div}(\omega)/2)$.

known results 2

Theorem (Boissy (2015))

Similar result as Kontsevich-Zorich in the case of meromorphic abelian differentials.

Theorem (Lanneau (2008) ; Chen-Möller(2014))

For $k = 2$ and at most simple poles, the strata of primitive quadratic differentials may have a hyperelliptic component or a sporadic component and are connected otherwise.

Theorem (Chen-G. (2021))

For $k = 2$ and at least one pole of higher order, the strata may have a hyperelliptic component and are connected otherwise.

Known results 3

Definition

A stratum of k-differentials $\Omega^k\mathcal{M}_g^{\text{prim}}(m_1,\ldots,m_n)$ *is said to be of* parity type *if the 2-adic valuation of every* m_i *is different from the 2-adic valuation of* k.

Proposition

Let (X,ξ) *be a k-differential in the stratum* $\Omega^k\mathcal{M}_g^{\text{prim}}(\mu)$. *Then the canonical cover of* (X,ξ) *has only even order singularities if and only if the 2-adic valuation of every entry of* μ *is not equal to* $v_2(k)$.

Known results 4

Theorem (Chen-G. (2021))

Let $S = \Omega^k\mathcal{M}_g^{\text{prim}}(\mu)$ be a stratum of primitive k-differentials of parity type with $g \geq 1$. If k is even or $S = \Omega^3\mathcal{M}_2^{\text{prim}}(6)$, then the parity is an invariant of the entire stratum S. If k is odd and $S \neq \Omega^3\mathcal{M}_2^{\text{prim}}(6)$, then there exist components of S with distinct parity invariants.

Idea of proof 1

Theorem (Bainbridge-Chen-G.-Grushevsky-Möller (2019) ; Costantini-Möller-Zachhuber(2019))

There is a smooth compactification of the (projectivised) strata of k-differentials. This compactification is the moduli space of the multi-scale k-differentials.

Idea of proof 2

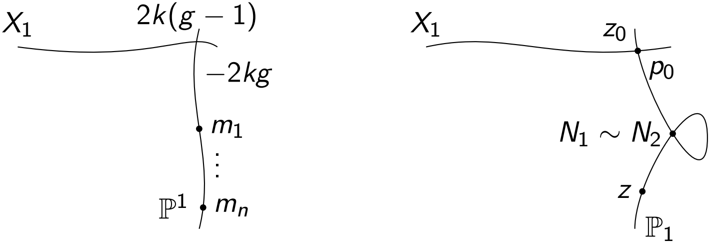


FIGURE – Two multi-scale k-differential.

Idea of proof 3

1. If the k-differential has more than one zero, then merge them to obtain a multi-scale k-differential pictured in the left of last page ;
2. If it has a unique zero, show that in any component of the stratum, there is a degeneration to the multi-scale differential pictured in the right of last page ;
3. Understand how the divisors given by the multi-scale k-differentials of point 2 intersect ;
4. Classify the connected components in genus 2 ;
5. Make the recurrence.

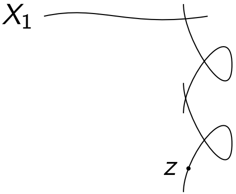
Idea of proof 4

How to find the degenerations in the moduli space of multi-scale differentials in point 1 and 2 in the case of abelian and quadratic differentials ?

- In the holomorphic case : find a special differential using dynamical systems.
- In the meromorphic case : use the flat representation given by gluing half planes.

Idea of proof 5

To see how the divisors intersect, we look at multi-scale k-differentials of the following form :



Finally, the components in genus two is clear in the case of abelian and quadratic differentials. We prove with Dawei that $\Omega^3\mathcal{M}_2^{\mathrm{prim}}(6)$ is connected.

¡Thanks for your attention !