Extending the DR cycle on $\overline{M}_{g,n}$ ...

Basic Q: $C, x_1, \ldots, x_n$ ~~smooth~~ marked curve, $k, a_1, \ldots, a_n \in \mathbb{Z}$, $\sum_i a_i = k(2g-2)$.

Q: When does $\exists$ a k-differential ω on C with $\operatorname{div}\omega = \sum_i a_i x_i$?

Smooth case: $\pi : C \to B$ smooth proper curve, conn. g. fibres,

$x_1, \ldots, x_n : B \to C$ disj. sections.

Line bdle $\mathcal{L} := \omega^{\otimes k}(-\sum_i a_i x_i)$ on C.

Naive def:

$$DR_{k,a_1 \ldots a_n} := \{ b \in B \mid \mathcal{L}|_{C_b} \simeq \mathcal{O}_{C_b} \} \subseteq B.$$

Claim $DR_{k,a_1 \ldots a_n} \subseteq B$ is closed, algebraic, & carries VFC of codim g.

Pf:

$$\begin{tikzcd}
\mathfrak{C}_{/B} \arrow[d] \\
B \arrow[u, bend left=50, "{[O]=e}"] \arrow[u, bend right=50, "{\sigma=[\mathcal{L}]}"']
\end{tikzcd}$$

locus

$$DRL_{k,a_1,\dots,a_n} = \sigma^* e = e^* \sigma.$$

closed in B.

$\mathfrak{C}$ is separated $/B$.

Cycle: $DRC_{k,a_1,\dots,a_n} = \sigma^![e] = e^![\sigma] \in A^g(B)$

$= CH^g(B).$

pullback of cycles a la Fulton

$\square$

Extend to (pre) stable curves.

$$\mathrm{DR}_{k,a_1,\ldots,a_n} \overset{\mathrm{def}}{:=} \{ b \in B \mid \mathcal{L}|_{C_b} \simeq \mathcal{O}_{C_b} \} \subseteq \overset{\overline{\mathcal{M}}_{g,n}}{\overset{\shortparallel}{B}}.$$

Problem: DR not closed in B (only loc. closed).

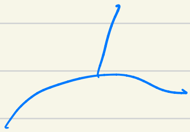
$$\begin{tikzcd}
\mathcal{M}_{g,n} \arrow[rr, "{\sigma = [\omega^{\otimes k}(\ldots)]}"] \arrow[dr] & & J_{g,n} = \mathrm{Pic}^{\underline{0}}_{C/\overline{\mathcal{M}}_{g,n}} \arrow[dl, shift left] \\
& \overline{\mathcal{M}}_{g,n} \arrow[ur, shift left, "{[\mathcal{O}] = e}"] \arrow[ur, dashed, bend right, "{\sigma = [\omega^{\otimes k}(\ldots)]}"'] &
\end{tikzcd}$$

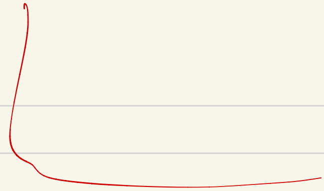
(multideg $\underline{0}$)

smooth, semiabelian, connected fibres.

Idea: blow up $\overline{\mathcal{M}}_{g,n}$ to resolve indeterminacy of σ.

"as large as possible"





$$\begin{tikzcd}
\mathcal{M}_{g,n}^{k,a_1,\dots,a_n} \arrow[r, "\bar\sigma"] \arrow[d, hook, "\text{open imm.}"'] & \mathrm{Pic}^0 \arrow[d] \\
\overline{\mathcal{M}}_{g,n}^{k,a_1,\dots,a_n} \arrow[r, "\text{log blowup}"'] & \overline{\mathcal{M}}_{g,n} \arrow[u, dashed, bend left, "\sigma"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Pic}^0_{\bar C/\mathrm{Pic}} \arrow[d] \\
\mathrm{Pic}^{[0]}_{C/\overline{\mathcal{M}}} \arrow[u, dashed, bend left, "\mathcal{L}"]
\end{tikzcd}$$

Eg. If $g=1$, $n=2$, $k=\dots$, $a_1=1$, $a_2=-1$,

$$\overline{\mathcal{M}}_{1,2}^{k,1,-1} = \overline{\mathcal{M}}_{1,2}, \quad \& \quad \mathcal{M}_{1,2}^{k,1,-1} = \overline{\mathcal{M}}_{1,2} \setminus \{\ \}.$$

Moral: we threw stuff away, but it didn't matter.

<u>Def</u>: $\widetilde{DRL}_{k,a_1,\dots,a_n} = \bar\sigma^* e \underset{\text{closed}}{\subseteq} \mathcal{M}_{g,n}^{k,a_1,\dots,a_n}$

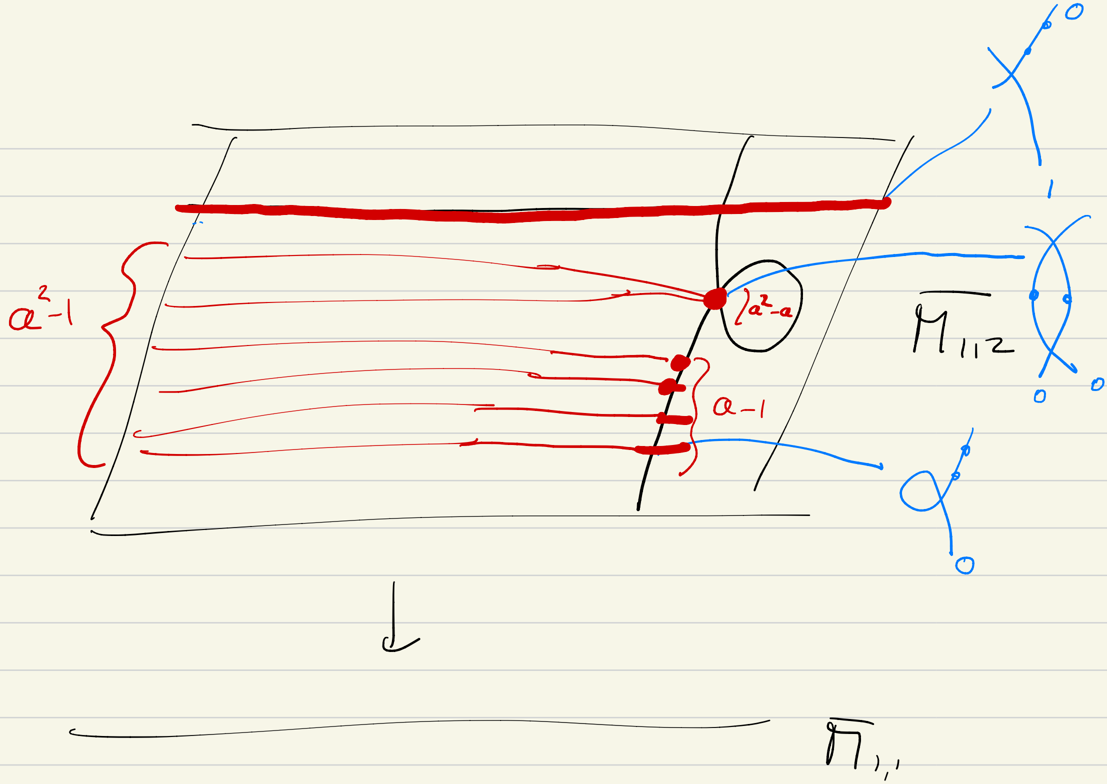
$\widetilde{DRC}_{k,a_1,\dots,a_n} = \bar\sigma^! [e]$ cycle supported on $\widetilde{DRL}_{k,a_1,\dots,a_n}$.

Thm: $\widehat{DRL}_{k,a_1,\dots,a_n} \xrightarrow{f} \overline{M}_{g,n}$ is proper.

Def. $\overline{DRC}_{k,a_1,\dots,a_n} := f_* \widetilde{DRC}_{k,a_1,\dots,a_n} \in A^g(\overline{M}_{g,n})$.

Examples.

$C/\overline{M}_{1,n}$. Choose $k=0$, ie $\omega^{\otimes 0} = \mathcal{O}$.

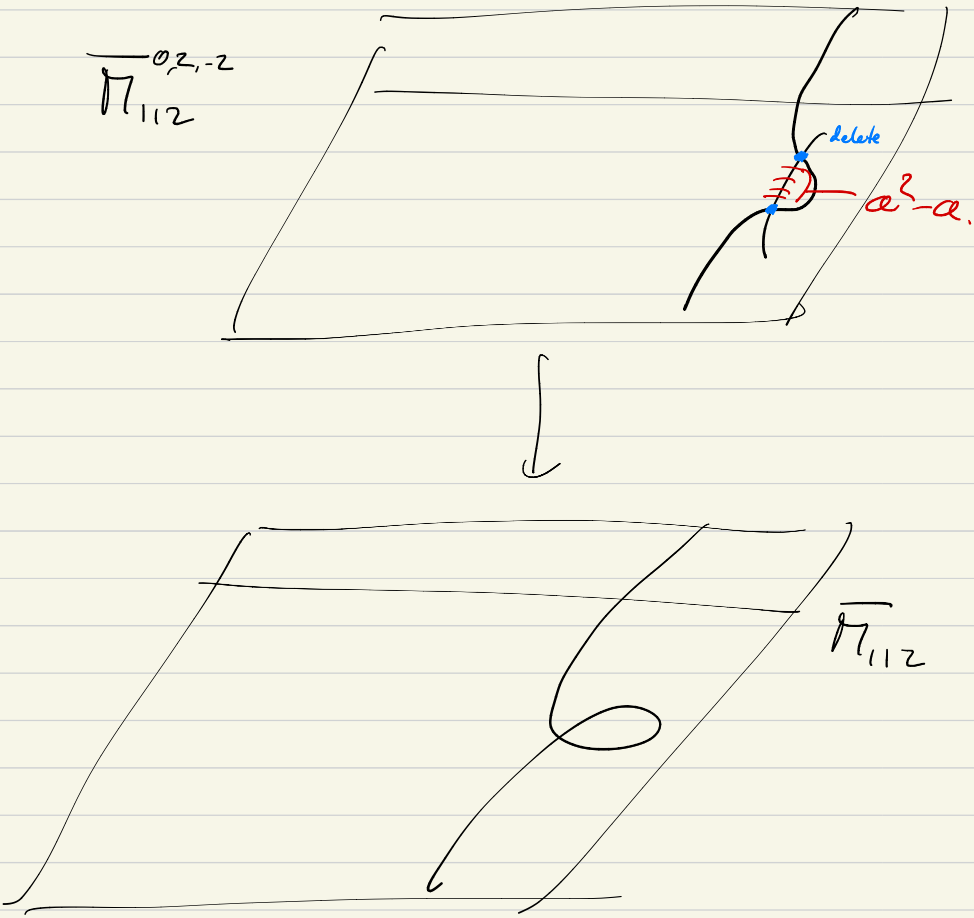


DR in $M_{1,2}$ $\quad k=0, \quad a_1 = a, \quad a_2 = -a.$

$\mathcal{L} = \mathcal{O}(a x_1 - a x_2).$

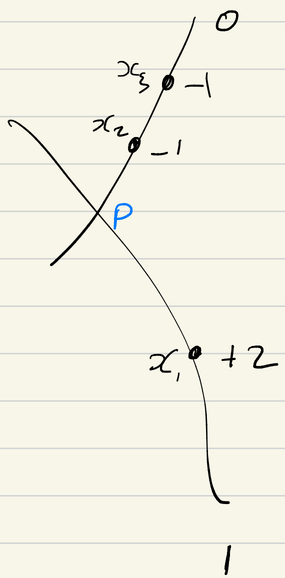
DR in $\overline{M}_{1,2}$

$a = 2.$



Tropical approach to choosing blowup/what to delete.

<u>First problem</u> $\overline{M}_{1,3}$ $\underline{a} = (2,-1,-1)$



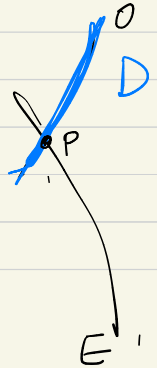
<u>Ex</u>; this lies in the closure of DR if & only if $p - x_1$ is 2-torsion.

i.e. $(\mathcal{O}(2p - 2x_1) \simeq \mathcal{O})$.

<u>But</u>. $\mathcal{O}(2x_1 - x_2 - x_3)$ is <u>never</u> trivial.

(degrees not $(0,0)$)

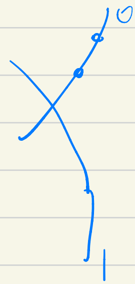
Idea to fix:



think of D as a divisor, & look at

$\mathcal{L}' = \mathcal{O}(\sum_i a_i x_i - 2D)$.

D divisor in $C/\overline{M}_{1,3}$, supported over

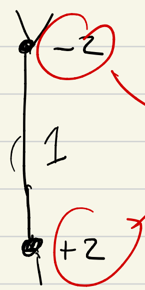


$$\mathcal{O}(D)\Big|_D \simeq \mathcal{O}_D(-P)$$

$$\mathcal{O}(D)\Big|_E \simeq \mathcal{O}_E(P)$$

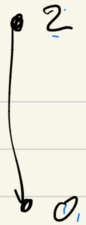
Twists in
Farkas–Pandharipande
BCCGM.

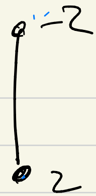
Tropical version



tropical divisor.

Q: Is the tropical divisor principal?
i.e. is it the divisor of some PL fctn?

$f =$ 

, $\operatorname{div} f$ 

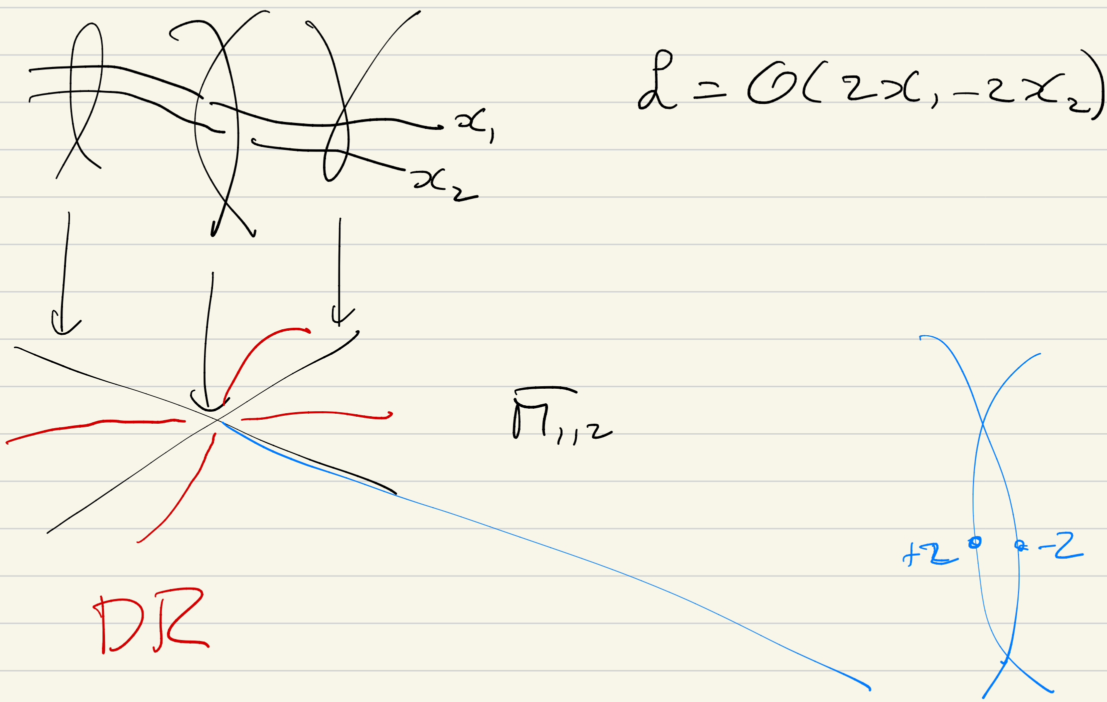
$$(\operatorname{div} f)(v) = \sum_{v \xleftarrow{e} u} \frac{f(u)-f(v)}{\underbrace{\ell(e)}_{=1}}$$

Tropical divisor $\to$ vertical divisor on curve

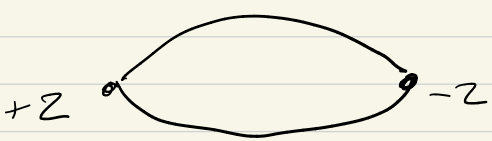
correcting to multideg $\underline{0}$

Second problem: $\overline{M}_{1,2}$ $\underline{a} = (2,-2)$.

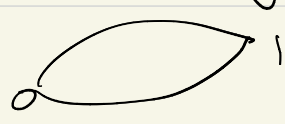
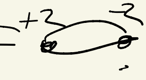
$\mathcal{L} = \mathcal{O}(2x_1 - 2x_2)$

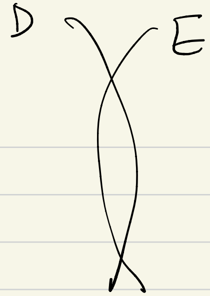


multideg $\neq \underline{0}$,
so cannot be
trivial bundle.



is naively principal,

$f =$  , div $f =$ 

Problem: 

are codim 2 in C

$\overline{M}_{1,2}$

not divisors.

Fix: - blow up $\overline{M}_{1,2}$ to fix dimension.

- or, do more careful tropical divisor, paying attention to edge lengths.

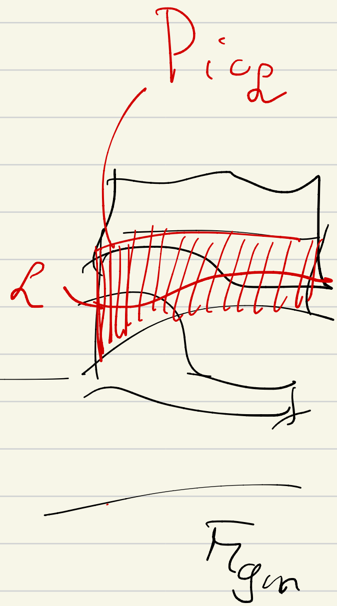
<u>M-W:</u> $\mathrm{Div}: (C/S, P, \alpha)$

C/S log curve,

P $\mathbb{G}_m^{\mathrm{trop}}$ torsor on S

$\alpha: C \to P$

$\underline{aj}: \mathrm{Div} \longrightarrow \mathrm{Pic}$

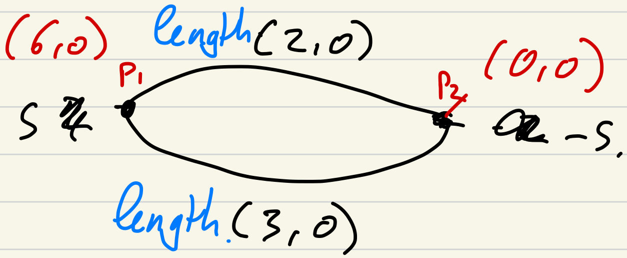


$\mathcal{L} = \omega^{\otimes k}(\ldots) : \overline{M}_{g,n} \longrightarrow \mathrm{Pic}$

$$\begin{array}{ccc} M_{g,n}^{k,a_1\ldots a_n} & \longrightarrow & \mathrm{Pic}_{\mathcal{L}} \\ \downarrow & & \downarrow \\ \mathrm{Div} & \xrightarrow{aj} & \mathrm{Pic} \end{array}$$

$$\mathrm{Div}(C/S) \approx \left(\begin{array}{c} \text{PL fctn on } C/S \\ \text{up to constants} \end{array} \right)$$

$$\Pi = \mathbb{N}^2$$



Q: is $(2,-2)$ principal?

$f: \mathrm{Vert} \to \mathbb{Z}^2$ PL because

$P_1 \longmapsto (6,0)$ $\quad$ $(6,0)-(0,0) = 3(2,0)$

$P_2 \longmapsto (0,0).$ $\quad$ $(6,0)-(0,0) = 2(3,0).$

$$(\mathrm{div} f)(P_1) = \sum_{P_1 \overset{e}{\text{---}} v} \frac{f(v) - f(P_1)}{\ell(e)} = \frac{(0,0)-(6,0)}{(3,0)} + \frac{(0,0)-(6,0)}{(2,0)}$$

$$= -2 - 3 = -5.$$

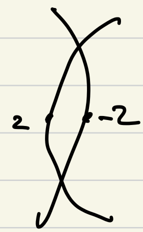
$$(\operatorname{div} f)(P_2) = +5.$$

So $(-5, 5)$ in principal.

Recall: $\overline{\mathcal{M}}_{1,2}$, $\underline{a}=(2,-2)$, section $\mathcal{L}=\mathcal{O}(2,-2)$.

Trying to define DR by intersecting with zero section of relative jacobian $\underline{\mathrm{Pic}}^{\underline{0}}$.

multidegree $\underline{0}$

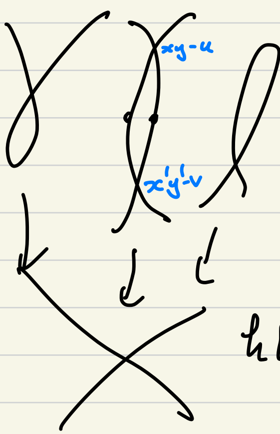
At banana point 

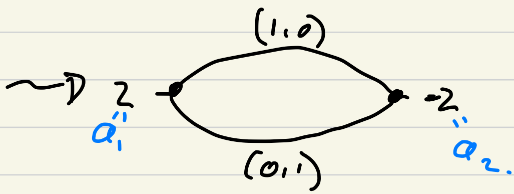
$\underset{\overset{\|}{(2,-2)}}{\text{multideg}} \neq \underline{a}$.

Would like to try to correct by twisting by vertical divisor, but dimensions wrong.

Fix: more careful tropical geometry: graphs with edge length in monoid $\mathbb{N}^2$.

$PL \subseteq \mathrm{Hom}(\mathrm{Vert}, \overline{M}^{gp})$





$k[u,v]=k[\mathbb{N}^2]$

$u \longrightarrow (1,0)$

$v \longrightarrow (0,1)$

$\mathrm{Div} = \mathrm{Hom}(\mathrm{Vert}, \mathbb{Z})$.

$f \in PL$, $\mathrm{div} f \in \mathrm{Div}$.

$f : \mathrm{Vert} \to \mathbb{Z}^2$ is PL iff.

for all edges $\underset{v_1}{\bullet} \xrightarrow{\ e\ } \underset{v_2}{}$, $\ell(e) \,\big|\, \big(f(v_1) - f(v_2)\big)$

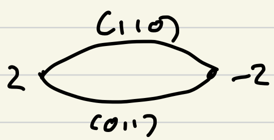
But if $(1,0) \mid f(v_1) - f(v_2)$ & $(0,1) \mid f(v_1) - f(v_2)$

then $f(v_1) = f(v_2)$.

So no non-constant PL functions.

Idea: change the monoid / logstr.

<u>Lem</u>: $\varphi : \mathbb{N}^2 \to P$ map of sharp fs monoids.

Then 

becomes principal over P

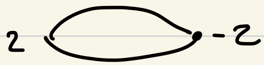
iff $\varphi(1,0) = \varphi(0,1)$ or at least one of

$\underline{\varphi(1,0),\ \varphi(0,1) \text{ is } 0 \in P}$.

Your map to $T_{1,1,2}$ missed the banana.

<u>Pf</u>. ex. $(\square)$

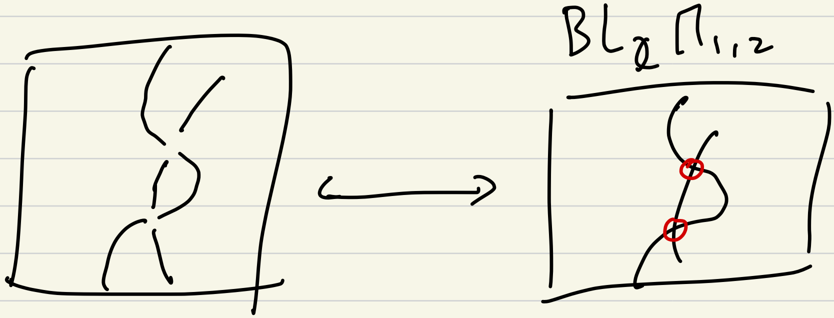
Cor: $t : T \to \overline{M}_{1,2}$ s.t. $t^{-1} M_{1,2}$ sch.-dense.

Then 

is tropically principal on T $\iff$ t^*u & t^*v differ by mult. by a unit in $\mathcal{O}_T$,

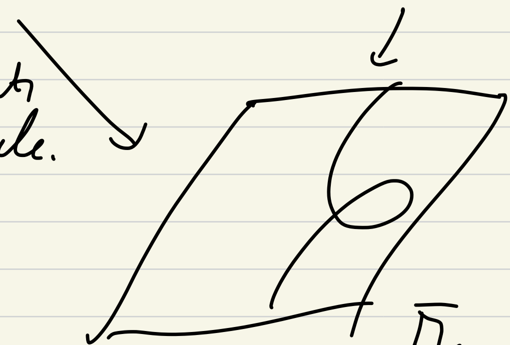
$\iff$ $t : T \to \overline{M}_{1,2}$ factors via open $M^{\alpha}_{1,2}$ in $Bl_{\ell}\, \overline{M}_{1,2}$ obtained by deleting codim-2 strata:

$M^{\alpha}_{1,2}$



$Bl_{\ell}\, \overline{M}_{1,2}$

birat'l log etale.



$\overline{M}_{1,2}$

$R = k[u,v]$

$A = \dfrac{k[u,v,s^{\pm 1}]}{(u - sv)}$ patch of blowup.

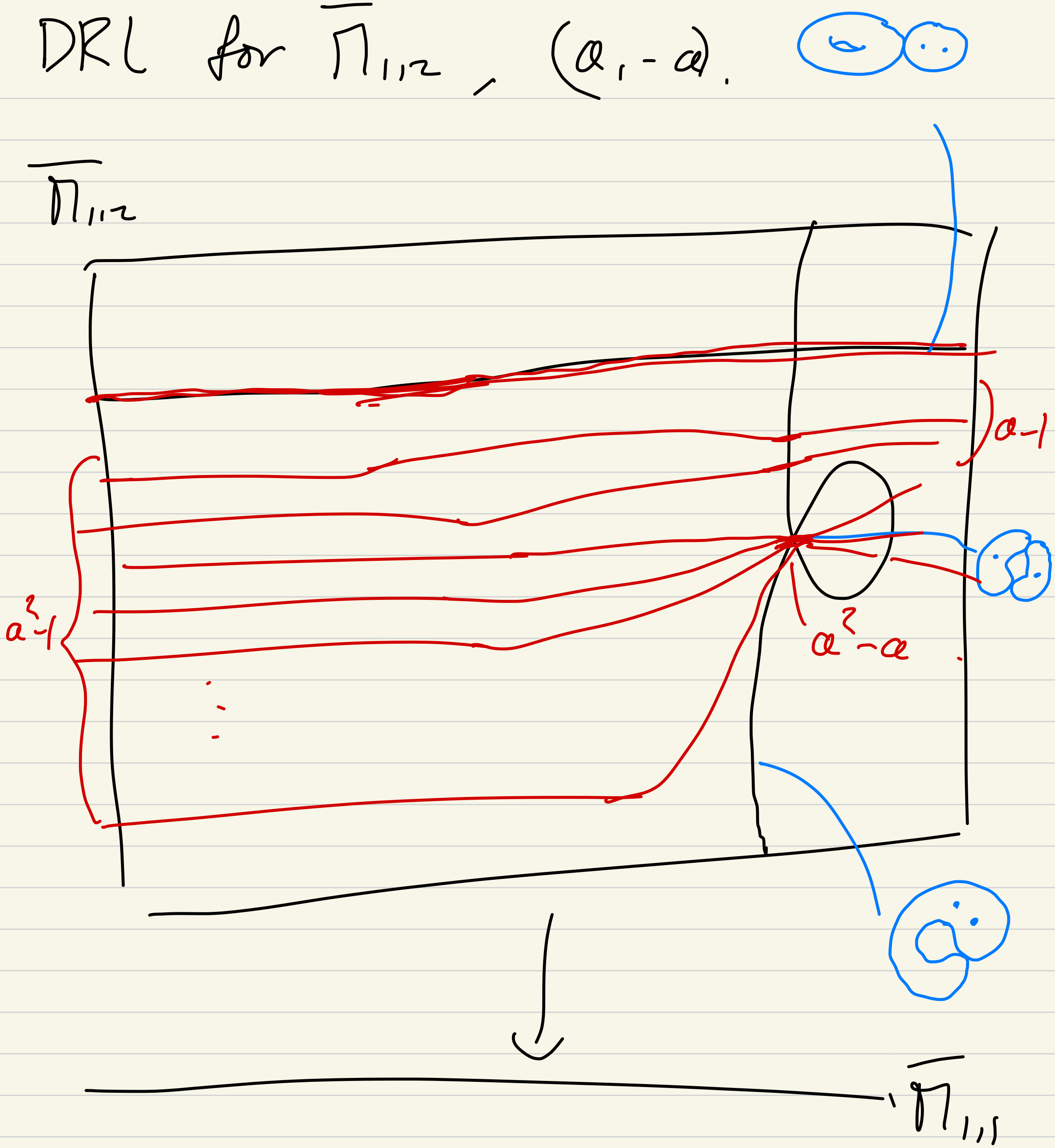
Define $\mathrm{DRL} \rightarrowtail \overline{\mathcal{M}}^{a}_{1,2}$ to be locus where the two sections of Pic^{0} intersect.

Lem: $\mathrm{DRL} \to \overline{\mathcal{M}}_{1,2}$ is proper.

$\rightsquigarrow$ define DR by pushforward.

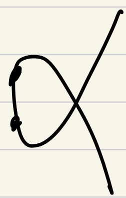
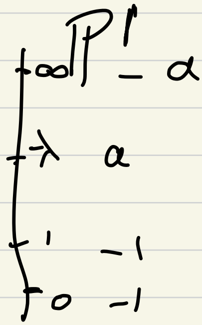
(works similarly $\forall\, g, n, k, \underline{a}$; harder to draw pictures).

DRL for $\overline{M}_{1,2}$, $(a, -a)$.



How do we know are $a-1$ ~~bte~~ pts. through  locus?

1) $a-1$ nontrivial torsion pts in $\mathbb{G}_m = k^*$.

2)  glue 0 & ~~∞~~ 1 $\longleftarrow$ 

BCCGM: $\overline{H}(a;a) =$ closure of $DR|_{M_{1,2}}$.

Diff form with divisor

$$w = (z+\lambda)^{a} z^{-1} (z-1)^{-1} dz.$$

BCCGM: deform the differential on smooth curve iff residues at 0 & 1 sum to 0.

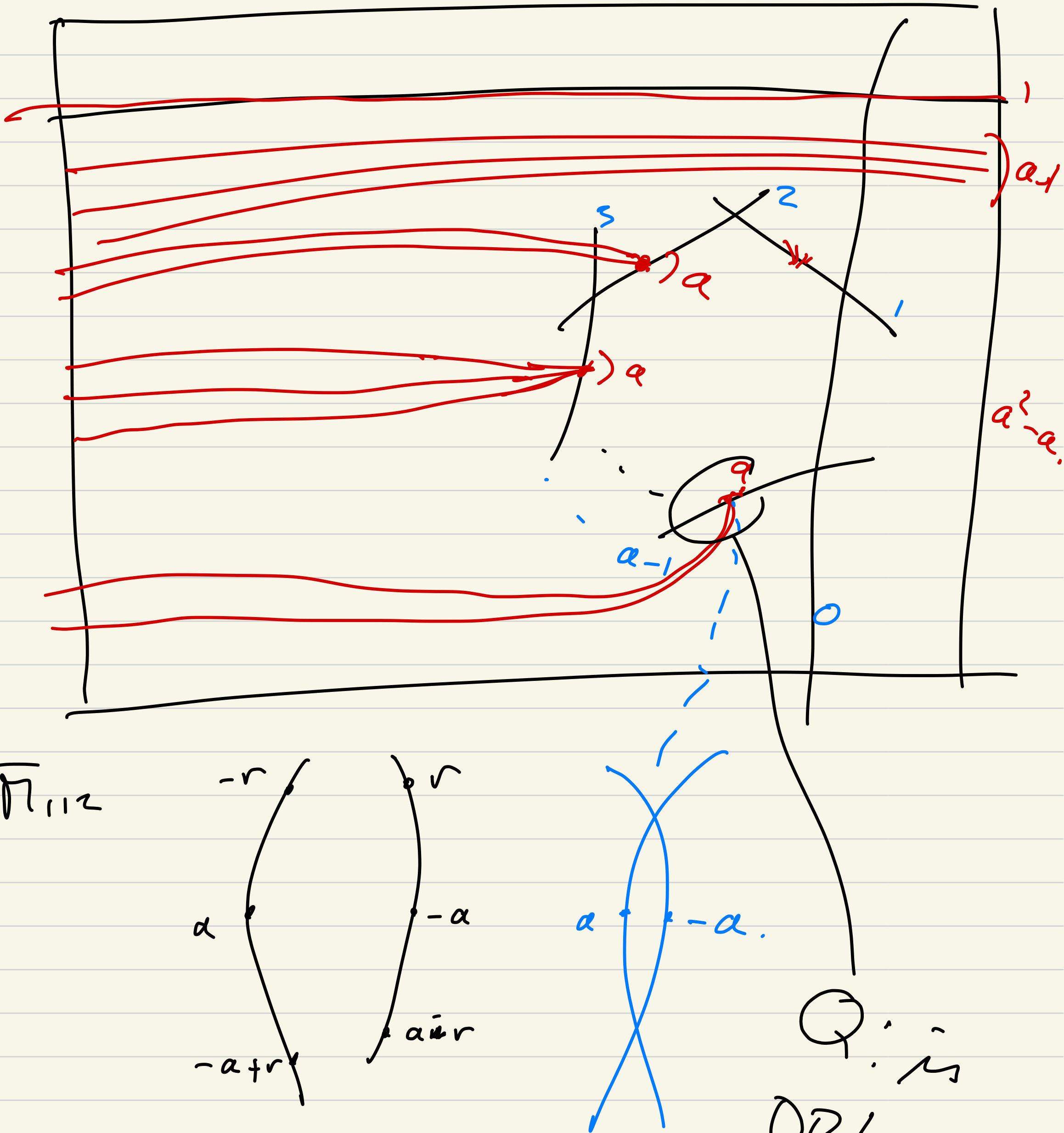
This gives

$$0 = (-\lambda)^a + (\lambda+1)^a = a\lambda^{a-1} + \cdots + 1.$$

So $a-1$ roots (distinct?)

(if $a \in h^{+}$).

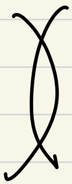
Π_{112}



Q: is ORL smooth here??

[H-Schmitt, pf of lem 5.5: ~~a pt~~

a differential on the boundary



deforms to one on a smooth curve.

$\Longleftrightarrow$ (sum of residues at glued pts $= 0$.)]