Rubber & Multiscale differentials

(with many people: S, W, CGGGΠ).

Discussing the papers:

[MaW]

[Marcus-Wise, log. cpct. of the Abel-Jacobi section (arxiv 2017, PLMS 2020, 34 pages)] [Please update arxiv version !].

[BCGGΠ], The moduli space of multiscale differentials (arxiv 2019, 122 pages)

Bainbridge, D. Chen, Gendron, Grushevsky, Möller.

$\mathcal{M}$ = stack of log/prestable curves (take $\overline{\Pi}_{g,n}$ if preferred)

$\mathcal{C}/\mathcal{M}$ univ. curve.

$Pic^{=Pic}_{\mathcal{C}/\mathcal{M}} \longrightarrow \mathcal{M}$ universal Picard; stack of line bdles.

Details to follow, but in brief:

[MaW]: space $\mathrm{Rub} \xrightarrow{aj} \mathrm{Pic}_{C/M}$, log blowup of Div, ndPR cycle etc. (Pixton formulas).

[BCGGM]: space $\mathcal{GMS}/\mathcal{M}$

Global res. cond. GRC cuts out $\mathcal{MS} \subseteq \mathcal{GMS}$,

$\mathcal{MS}$ is 'main component': smooth stack, locus of smooth curves is dense.

Today, recall defs then sketch isomorphism

$$\mathrm{Rub} \times \mathcal{M} \xrightarrow{\sim} \mathcal{GMS}.$$

aj, Pic, w

(Goal is really to set up a dictionary between these worlds).

$$\begin{array}{ccc} \mathcal{Q} & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow{\scriptstyle (C/S,\, \omega_{C/S})} \\ \mathrm{Rub} & \xrightarrow[aj]{} & \mathrm{Pic} \end{array}$$

We had 3 hrs of talks in spring giving careful def. of GMS & SS (2 × DC, 1 × Höllen).

• 2 hrs from Jonathan explaining Rub.

So I clearly can't recall all of these in detail, but I also don't want to assume you remember them perfectly.

Plan: - Briefly sketch def of Rub & aj : Rub → Pic ;

- show how to extract the various pieces of data needed for GMS.

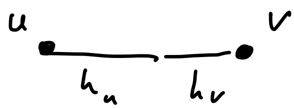
As with all talks, this is an exercise in guessing how much my audience already know/remember, but I think this one is a bit harder than usual. So please do not hesitate to ask if I skip over things you would like to be reminded of, or if I go too slowly through things that are clear to everyone.

Def [MaW + ε²] Rub is the fibred Cat over $\mathfrak{M}$ with objects $(C/S, \beta \in \overline{M}^{gp}_C(C))$ where

C/S log curve, satisfying two conditions;

Notation

∀ $\bar{s}$ geom. pt of S, we can see β_s as a PL function on the tropicalisation $\mathfrak{X}_s$; in other words, β_s assigns to each vertex an elt of $\overline{M}^{gp}_{S,\bar{s}}$, & to each half-edge a slope $\in \mathbb{Z}$, s.t. ∀ edges



- $\beta_s(h_u) = -\beta_s(h_v)$
- $\beta_s(v) - \beta_s(u) = \mathrm{len}(h_u) \cdot \beta_s(h_u)$

(change in height = length × slope)

1) We require Image of β_s on {vertices} is totally ordered with largest elt. I in $\overline{M}^{gp}_{S,\bar{s}}$.

($\varepsilon^2$2) The fibre product

$$\begin{array}{ccc} \bigcirc & \longrightarrow & Bl_{Im(\beta)}[\mathbb{P}^1/\mathbb{G}_m] \\ \downarrow & & \downarrow \\ C & \xrightarrow{\beta} & \mathbb{G}_m^{trop} \end{array}$$

is a log curve.

Rk on cond. 2

Cond 2 won't appear so much in what follows, hence only sketched def. But a couple of remarks:

- Was overlooked in [MaW], but will be in forthcoming paper of Jonathan, Rahul, (Sam?) (needed to make [MaW] theorems true.

- With Cond 2, Rub is a smooth stochastic process classically (answers Q of Rahul from spring).

- Cond 2 is ~ the difference between 'simple' & 'non-simple' constructions in BCGGM.

BC GGΠ data:

Let S s.h. local atomic, closed pts $\in S$.

$(C/S, \beta \in \Pi_C^{gp}(C))$ pt of Rab.

BCGGΠ language

Def. The **enhanced level graph** is the data of

- graph $\mathcal{X}_s$
- the total preorder on vertices of $\mathcal{X}_s$ induced by β;
- the slopes $\beta(h) =: \kappa_e$.

$$\beta(h) = \frac{\beta(u) - \beta(v)}{\text{length}(e)}.$$

Def A **log splitting** is a section of $\Pi_S \to \overline{\Pi}_S$

The **simple level rotation torsor** is the set of log splittings.

The simple level rotation group is $\mathrm{Hom}_{\mathrm{Mon}}(\Pi_S, \mathcal{O}_S^\times)$.

Easy lem: The simple level-stable torsor is a torsor under the simple level rot. gp.

Prong matchings

Given edge e of $\mathcal{X}_S$, let $Z_e \to S$ be locus cut out by $\mathrm{len}(e) \in \Pi_S$ (i.e. locus where node persists).

Then if $\widetilde{X} \to X_{Z_e}$ is (non-log!) blowup along section induced by e, let $\tilde{e}$ = preimage of e in $\widetilde{X}$, fin. etale deg 2 over Z_e, $i: \tilde{e} \to \widetilde{X}$. Define

$$\pi \searrow \quad \swarrow Z_e$$

$$\mathcal{N}_e = \det \pi_* i^* \omega_{\widetilde{X}/Z_e}$$

- this is just the pullback of the normal bundle of the appropriate boundary divisor in $\mathcal{M}$.

Lem: In above notation, there is a canonical iso

$$N_e \xrightarrow{\sim} \mathcal{O}_{\mathbb{Z}_e}(\mathrm{len}(e)).$$

Pf. Canonical logstr. on M is divisorial Lam boundary, & both sides commute w. pullback. □

Def. A **local prong matching** at edge e is Lam section of N_e.

Rk. A choice of log splitting $\overline{\Pi}_S \xrightarrow{P} \Pi_S$ determines a local prong matchings at all edges e:

$$\begin{array}{ccc} \overline{\Pi}_S & \xrightarrow{P} & \Pi_S \\ & & \cup | \\ & & \mathcal{O}(\mathrm{len}(e)) \xrightarrow{\sim} N_e \\ & & \text{by prev. lem.} \\ \mathrm{len}(e) & \longmapsto & P(\mathrm{len}(e)) \end{array}$$

Differentials & AJ map

Want a gen. multiscale differential, but no differentials in the picture yet.

The Abel-Jacobi map $\mathrm{Rub} \xrightarrow{aj} \mathrm{Pic}$

$$(C/S, \beta \in \bar{\Pi}_C^{gp}) \longmapsto (C/S, \mathcal{O}_C(\beta))$$

$$1 \to \mathcal{O}_C^\times \to \Pi_C^{gp} \to \bar{\Pi}_C^{gp} \to 1$$

β,

pretmage is $\mathcal{O}_C^\times(\beta)$, $\mathcal{U}$ in $\mathcal{O}$.

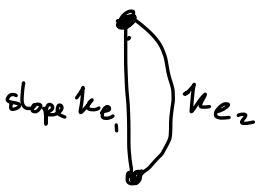
So a pt of $\mathrm{Rub} \times_{aj, \mathrm{Pic}, \omega} \mathcal{M}$ is a tripple

$$(C/S, \beta \in \bar{\Pi}_C^{gp}(C), \mathcal{O}_C(\beta) \xrightarrow{\sim} \omega_{C/S}).$$

To extract a collection of rescaled differentials:

- β takes max. value 1; on those vertices $\mathcal{O}_C(\beta)$ is canonically trivial, so induces a section of $\omega_{C/S}$.

- on next level down;



$\beta = 0 \quad (\text{on } I \dots)$

$\beta = -k_{e_1} \cdot \text{len}(e_1) \quad = -k_{e_2} \cdot \text{len}(e_2).$

Log splitting $\overline{M}_S \xrightarrow{P} N_S$ determines sections of

$\mathcal{O}(\text{len}(e_1))$ & $\mathcal{O}(\text{len}(e_2))$, determining

same section of $\mathcal{O}(\beta) = \mathcal{O}(\text{len}(e_1))^{\otimes -k_{e_1}}$

$= \mathcal{O}(\text{len}(e_2))^{\otimes -k_{e_2}}$

because p is a monoid hom.

These then determine ~~the~~ differential on next level down, etc.

The choice of log splitting $\overline{\Pi}_S \longrightarrow \Pi_S$ determines everything in sight, & is acted on by simple log level rot.-gp. $\mathrm{Hom}(\overline{\Pi}_S, \mathcal{O}_S^\times)$.

Quotient in pf of GΠS.