Moduli spaces of admissible covers and the $\mathcal{H}$-tautological ring

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Moduli of branched covers

- Let $\mathcal{H}_{g/h,d}$ be the **Hurwitz space**

$$\mathcal{H}_{g/h,d} = \left\{ \begin{array}{l} f : X \to Y \\ X, Y \text{ smooth, connected, proper curves of genus } g, h \\ f \text{ simply ramified of degree } d \end{array} \right\}$$

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- We have **source** and **target** maps:

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- In particular, $\mathcal{H}_{g/h,d}$ is a smooth DM stack of dimension $3h - 3 + b$.
- On the other hand, $\phi_*([\mathcal{H}_{g/h,d}]) \in A_{3h-3+b}(\mathcal{M}_g)$ is an interesting algebraic cycle.

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Goal: explain compactification of $\overline{\mathcal{H}}_{g/h,d}$ by **admissible covers** (with variations) and *intersection-theoretic aspects*.

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- *Birational geometry.*
  1. $\overline{\mathcal{M}}_{g\geq 24}$ is of *general type*, via locus $\phi_*([\overline{\mathcal{H}}_{g/0,\frac{g+1}{2}}]) \in A^1(\overline{\mathcal{M}}_g)$ of curves of sub-maximal gonality (Eisenbud, Harris, Mumford)
  2. *Extremal cycles* on $\overline{\mathcal{M}}_{g,n}$ (Blankers, Chen, Coskun, Patel, Tarasca...)

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- *Numerological phenomena.* E.g.:

Conjecture (L., 2020)

For any $g \geq 2$,

$$\sum_{d\geq 1} \phi_*([\overline{\mathcal{H}}_{g/1,d}])q^d \in A_{2g-2}(\overline{\mathcal{M}}_g) \otimes \mathbb{Q}[[q]]$$

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- *Non-tautological classes.* (Graber-Pandharipande, Petersen-Tommasi, van Zelm) More later.

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Definition (Harris-Mumford, 1982)

Let $(Y, y_1, \ldots, y_b)$ be a stable curve. Then, an **admissible cover** is a finite morphism $f : X \to Y$ such that:

- X is a connected nodal curve,
- f is étale over over $Y^{\mathrm{sm}} - \{y_1, \ldots, y_b\}$ and simply ramified over the y_i,
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- at each node of X, the ramification indices of f restricted to the two branches are *equal*.

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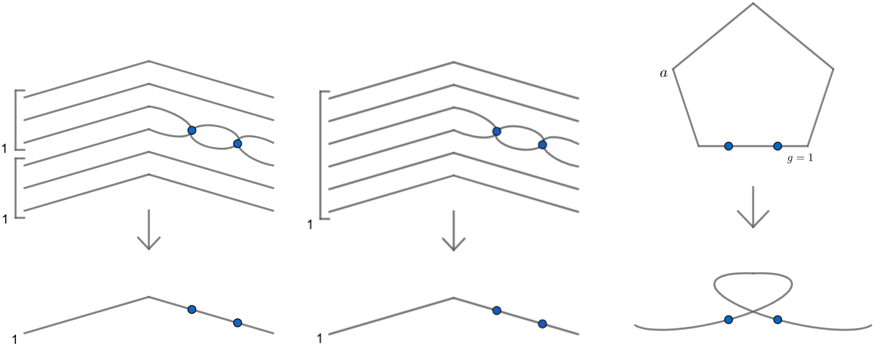
Over nodes, the complete local ring of the map induced by $f : X \to Y$ deformation spaces looks like

$$\mathbb{C}[[t_1, t_2]]/(t_1 t_2) \to \mathbb{C}[[s_1, s_2]]/(s_1 s_2)$$
$$t_1 \mapsto s_1^e$$
$$t_2 \mapsto s_2^e.$$

Examples of admissible covers

In all three examples, $g = 2$ and $h = 1$.



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- $\overline{\mathcal{H}}_{g/h,d}$ is singular in general!

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- For example, if $h = 0$, and λ, μ are partitions of d, then one can consider the space $\overline{\mathcal{H}}_{g/0,\lambda,\mu}$ of admissible covers with ramification profiles λ, μ over $y_1, y_2 \in Y$ and simply ramified everywhere else.

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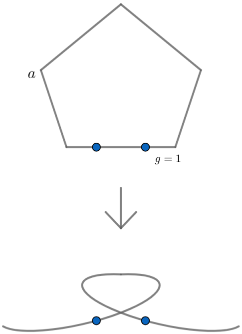
- The open part $\mathcal{H}_{g/0,\lambda,\mu}$, this is the moduli of meromophic functions ("0-differentials") $f : X \to \mathbb{P}^1$ with zeroes and poles of prescribed orders.

Failure of smoothness



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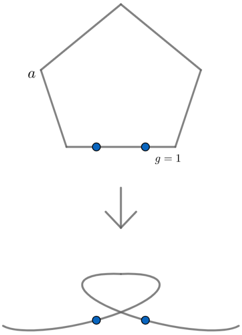

The complete local ring of this cover is

$$\mathbb{C}[[s, t, t_1, \cdots, t_m]]/(t = t_i^a)$$

where s is a deformation parameter "moving the branch points apart" (coming from $\overline{\mathcal{M}}_{0,4}$), and the t_i, t are deformation parameters of the nodes upstairs and downstairs, respectively.

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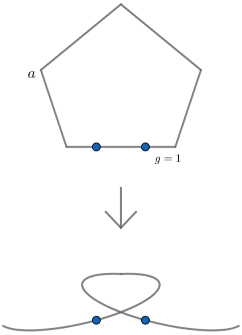

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- The deformations of $f : X \to Y$ are "almost" controlled by deformations of $(Y, y_1, \ldots, y_b)$, but for every pre-image of a node of Y with ramification index e, adjoin an e-th root of the corresponding smoothing parameter.

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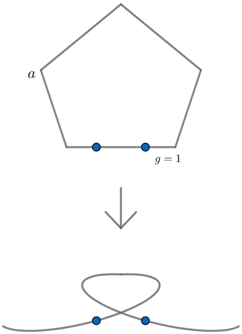

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- $\overline{\mathcal{H}}_{g/h,d}$ is Cohen-Macaulay, but fails to be smooth at covers with multiple ramification points over the same node.

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- By restricting to the étale loci on both sides, this extends to a correspondence

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- The correspondence breaks down when one allows f to degenerate to an admissible cover: there may be multiple ways to extend the map $\widetilde{f} : \widetilde{X}^{\text{sm}} \to Y^{\text{sm}}$ over the nodes.
- However, we do still have a map in the other direction: given if $\widetilde{f} : \widetilde{X} \to Y$ is an *admissible S_d-cover*, then $f : \widetilde{X}/S_{d-1} \to Y$ is an admissible cover of degree d. This will provide the desired normalization.

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Definition

Let G be a finite group. Then, an **admissible G-cover** $f : X \to Y$ is an admissible cover exhibiting X as a Galois cover of Y with Galois group G, such that at every node $x \in X$, the stabilizer $G_x \subset G$ is a cyclic group with generator g_x acting on the complete local ring $\mathbb{C}[[t_1, t_2]]/(t_1 t_2)$ at x by:

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- The moduli space of *pointed* admissible G-covers with source genus g and *monodromy data* ξ is denoted by $\overline{\mathcal{H}}_{g,G,\xi}$.
- Alternatively, $\overline{\mathcal{H}}_{g,G,\xi}$ is the space of curves with "admissible G-action."

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- On the other hand, ϕ is unramified (deformations as a G-curve inject into deformations as a curve).

Normalizing the Harris-Mumford space

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Theorem (Abramovich-Corti-Vistoli 2003*)

The moduli space $\overline{\mathcal{H}}_{\widetilde{g},S_d,\xi}$ of admissible S_d-covers provides a normalization

$$\nu : \overline{\mathcal{H}}_{\widetilde{g},S_d,\xi} \to \overline{\mathcal{H}}_{g/h,d}$$

of the Harris-Mumford stack, where $\nu([\widetilde{X} \to Y]) = [\widetilde{X}/S_{d-1} \to Y]$.

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$$\nu : \overline{\mathcal{H}}_{\widetilde{g},S_d,\xi} \to \overline{\mathcal{H}}_{g/h,d}$$

of the Harris-Mumford stack, where $\nu([\widetilde{X} \to Y]) = [\widetilde{X}/S_{d-1} \to Y]$.

In particular, we may replace the classes of the Hurwitz loci

$$\phi : \overline{\mathcal{H}}_{g/h,d} \to \overline{\mathcal{M}}_g$$

with classes of *Galois* Hurwitz loci

$$\widetilde{\phi} : \overline{\mathcal{H}}_{\widetilde{g},S_d,\xi} \to \overline{\mathcal{M}}_g,$$

where $\widetilde{\phi}([\widetilde{X} \to Y]) = \widetilde{X}/S_{d-1}$.

Twisted admissible covers and stable maps

Twisted admissible covers and stable maps

- Abramovich-Corti-Vistoli actually work in the setting of *twisted G-covers*. Instead of Galois covers $f : X \to Y$, where Y is isomorphic to the scheme-theoretic quotient $Y = X/G$, they consider maps

$$f' : X \to \mathcal{Y},$$

where f' is (globally!) a principal G-bundle over the *twisted curve* $\mathcal{Y}$, which is obtained from Y by adding *gerby structure* at the nodes and marked points.

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- In this way, the moduli space of admissible G-covers $\overline{\mathcal{H}}_{g,G,\xi}$ is essentially isomorphic to the space of *twisted stable maps* $\overline{\mathcal{M}}_{h,b}(BG, 0)$.
- We will prefer to work in the language of spaces $\overline{\mathcal{H}}_{g,G,\xi}$ of scheme-theoretic G-covers, which is more amenable to intersection theory.

Tautological classes on $\overline{\mathcal{M}}_{g,n}$

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Definition

The **tautological ring** is the minimal system of subrings

$$R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n})$$

containing ψ **and** κ **classes** and closed under by pushforwards by:

- **forgetful morphisms** $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$, and
- **boundary morphisms** $\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}$, where Γ is a **stable graph**.

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Theorem (Graber-Pandharipande, 2001)

The tautological ring is additively generated by **decorated boundary classes**, *that is, pushforwards of monomials in ψ and κ classes by boundary morphisms.*

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The proof is *constructive*: explicit algorithms for intersecting such classes are given. For example, if A, B are stable graphs, then:

$$\xi_A^*(\xi_{B*}([\overline{\mathcal{M}}_B])) = \sum_{\substack{\Gamma \text{ generic} \\ (A,B)\text{-graph}}} (\xi_{\Gamma \to A})_* \left(\prod_{(\ell,\ell') \in E(A) \cap E(B)} (-\psi_\ell - \psi_{\ell'}) \right),$$

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This is an application of the excess intersection formula; it is crucial that the spaces in question are smooth!

Non-tautological classes from branched cover loci

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Proposition (Graber-Pandharipande, 2001)

For all sufficiently large h, the class $\phi_*([\overline{\mathcal{H}}_{2h/h,2}]) \in A_{3h-1}(\overline{\mathcal{M}}_{2h})$ of genus $2h$ curves admitting a 2-to-1 cover of a genus h curve is non-tautological.

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- Proof idea: pull $\phi_*([\overline{\mathcal{H}}_{2h/h,2}])$ back to the boundary divisor $\xi_h : \overline{\mathcal{M}}_{h,1} \times \overline{\mathcal{M}}_{h,1} \to \overline{\mathcal{M}}_{2h}$.

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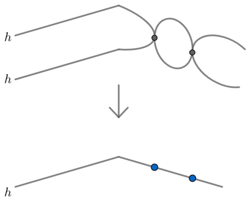

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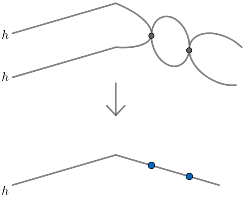

- Thus, $\xi_h^*(\phi_*([\overline{\mathcal{H}}_{2h/h,2}]))$ is a positive multiple of the diagonal $\Delta : \overline{\mathcal{M}}_{h,1} \to \overline{\mathcal{M}}_{h,1} \times \overline{\mathcal{M}}_{h,1}$.

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- If $\phi_*([\overline{\mathcal{H}}_{2h/h,2}])$, were tautological, its pullback $c \cdot [\Delta]$ to $\overline{\mathcal{M}}_{h,1} \times \overline{\mathcal{M}}_{h,1}$ would have a *tautological Künneth decomposition* in cohomology.

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- However, if $h >> 0$, then $\overline{\mathcal{M}}_{h,1}$ has *odd cohomology* (Pikaart), so such a Künneth decomposition of the diagonal class Δ can't exist.

The $\mathcal{H}$-tautological ring

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Goal: develop a theory of tautological classes on moduli spaces $\overline{\mathcal{H}}_{g,G,\xi}$ of admissible covers, that includes the classes of Harris-Mumford loci.

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As before, we have:

- ψ and κ classes: defined by pullback by $\phi : \overline{\mathcal{H}}_{g,G,\xi} \to \overline{\mathcal{M}}_{g,n}$,
- forgetful morphisms $\pi : \overline{\mathcal{H}}_{g,G,\xi+1} \to \overline{\mathcal{H}}_{g,G,\xi}$ (forgetting a G-orbit of unramified points on the source),
- boundary morphisms $\xi_{(\Gamma,G)} : \overline{\mathcal{H}}_{(\Gamma,G)} \to \overline{\mathcal{H}}_{g,G,\xi}$ where (Γ, G) is an **admissible G-graph**.

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New feature: **restriction** and **corestriction**

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- If $G_1 \subset G$, then one can restrict the action of G on a stable curve X to G_1. We thus define a map

$$\text{res}^G_{G_1} : \overline{\mathcal{H}}_{g,G,\xi} \to \overline{\mathcal{H}}_{g,G_1,\xi'}$$

by $\text{res}^G_{G_1}([X \to X/G]) = [X \to X/G_1]$.

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- If $G_1 \subset G$ is *normal*, then one can co-restrict the action of G on a stable curve X to an action of G/G_1 on X/G_1. We thus define a map

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- $\mathrm{res}^{G}_{\{1\}}$ recovers the source map ϕ.
- $\mathrm{cores}^{G}_{\{1\}}$ recovers the target map δ.
- Let $\overline{\mathcal{H}}_{\widetilde{g},S_d,\xi} = \widetilde{H}_{g/h,d}$ be the normalization of the Harris-Mumford stack. Then, we have a commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{H}}_{\widetilde{g},S_d,\xi} & \xrightarrow{\nu} & \overline{\mathcal{H}}_{g/h,d} \\ {\scriptstyle \mathrm{res}^{S_d}_{S_{d-1}}}\downarrow & & \downarrow{\scriptstyle \phi} \\ \overline{\mathcal{H}}_{\widetilde{g},S_{d-1},\xi'} & \xrightarrow{\delta} & \overline{\mathcal{M}}_g \end{array}$$

So the class of a Harris-Mumford locus is equal to that of a restriction-corestriction map.

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- diagonal morphisms $\Delta : \overline{\mathcal{H}}_{g,G,\xi} \to \overline{\mathcal{H}}_{g,G,\xi} \times \overline{\mathcal{H}}_{g,G,\xi}$.

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Definition

The $\mathcal{H}$-**tautological ring** is the minimal system of subrings

$$R^*_{\mathcal{H}}\left(\prod \overline{\mathcal{H}}_{g_i,G_i,\xi_i}\right) \subset A^*\left(\prod \overline{\mathcal{H}}_{g_i,G_i,\xi_i}\right)$$

inside the Chow rings of *products of* moduli spaces of admissible G-covers, containing all ψ and κ classes and closed under pushforwards by all (products of) forgetful, boundary, restriction-corestriction, and diagonal morphisms.

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Theorem (L., arXiv 2011.11565)

The $\mathcal{H}$-tautological ring is additively generated by pushforwards of monomials in ψ and κ classes under arbitrary compositions of tautological morphisms (which can be taken in a particular order).

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     In particular, we get a (reasonably efficient) method of integrating (non-Galois) Harris-Mumford loci against classical tautological classes.
  2. Intersections of restriction-corestriction morphisms with each other. Here the formulas depend on complicated group-theoretic data which may be difficult to compute in practice.

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- Study structure of $\mathcal{H}$-tautological ring, e.g. following Faber conjectures
- Look for (non-trivial) $\mathcal{H}$-tautological relations
- Look for non-$\mathcal{H}$-tautological algebraic cycles (or any obstructions to being $\mathcal{H}$-tautological)

Herzlichen Dank für Ihre Aufmerksamkeit!
Thank you for your attention!