Desingularising genus 1 moduli

Two key ideas:

curves w/ elliptic singularities ① $\longleftrightarrow$ log blowups induced by tropical order relations ②

Ranganathan–Santos-Parker–Wise I ('17)
absolute target.

$\downarrow$

- RSPW II ('17) relative toric boundary
  tropical realisability.
- Battistella–N.–Ranganathan ('19)
  relative smooth divisor
  recursive description of boundary / recursive algorithm
- Bozlee ('19): higher genus curves
- Battistella–Carocci ('20): $g=2$ maps.
- N.–Ranganathan ('19): Maximal contact log GW theory ($g=0$).

1) $\overline{M}_{1,n}(\mathbb{P}^r, d)$

- $\overline{M}_{1,n}(\mathbb{P}^r, d)$ has multiple irred. components.

- Main component: closure of locus where source curve is smooth; has $\dim = \operatorname{vdim} \overline{M}_{1,n}(\mathbb{P}^r, d)$.

- Excess components of $\overline{M}_{1,n}(\mathbb{P}^r, d)$:

$\boxed{\text{Excess dim} = r - k.}$

- Hope: isolate contribution of main component to GW invariants (reduced GW theory).

- Problem: main component not even virtually smooth.
- Goal: resolve singularities of main component.
- Already done:
  (Li-)Vakil-Zinger
  Hu-Li-Niu
- Today: alternative perspective.

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2) Curves w/ elliptic singularities

- What causes singularities of $\overline{M}_{1,n}(\mathbb{P}^r,d)$?

- Map $\overline{M}_{1,n}(\mathbb{P}^r, d) \longrightarrow \overline{M}_{1,n}$ is **obstructed** at points where $H^1(C, f^* T_{\mathbb{P}^r}) \neq 0$.

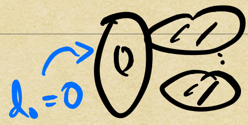
- Euler sequence for $T_{\mathbb{P}^r}$:

$$H^1(C, f^* T_{\mathbb{P}^r}) \neq 0 \iff H^1(C, f^* \mathcal{O}_{\mathbb{P}^r}(1)) \neq 0$$

$\iff$ the minimal genus one subcurve $E \subseteq C$ ("the core") is contracted by f.

- Main cpt: generically unobstructed

Excess cpt: obstructed everywhere.



$h^1(C, f^* \mathcal{O}(1)) = r.$

- Note: this is the same thing that causes quantum Lefschetz to fail for $g \geq 1$.

- Idea: change moduli problem, by disallowing contracted elliptic cpts.

- Need to replace by something:

$\boxed{\text{Gorenstein curve singularities of genus one.}}$

Smyth '08

Alternative modular compactification of $M_{1,n}$, allowing curves with "worse" singularities.

- Let $P \in C$ be an isolated singularity of a curve, and

$$\nu : \tilde{C} \to C$$

the normalisation at P.

Define invariants of singularity:

- $m = \# \nu^{-1}(P)$

  \# branches of C at P.

- $\delta = \dim_k \left(\nu_* \mathcal{O}_{\tilde{C}} / \mathcal{O}_C |_P \right)$

  \# conditions for a function on $\tilde{C}$ to descend to C.

Then define:

$$\boxed{g = g(C,P) = \delta - (m-1).}$$

conditions to descend, beside the obvious ones.

<u>E.g.</u>: $C = V(xy)$ nodal curve.

$m = 2$

$\delta = 1$ $\qquad \underset{\mathcal{O}_C}{\dfrac{k[x,y]}{xy}} \hookrightarrow \underset{\mathcal{O}_{\tilde{C}}}{k[x] \oplus k[y]}$

$\Rightarrow g = \delta - (m-1) = 0.$

Another way to say this:

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r, hook] \arrow[d, hook] & \mathbb{A}^1_y \arrow[d] \\
\mathbb{A}^1_x \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & V(xy)
\end{tikzcd}$$

$\underline{\text{E.g}}$: C = union of n co-ordinate lines in $\mathbb{A}^n$
"rational n-fold point"

$m = n$, $\delta = n-1$

$\Rightarrow g = 0$.

Why "genus"? If smoothen and apply semistable reduction, central fibre has nodal curve of genus g in place of singularity

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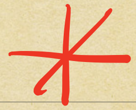
Theorem (Smyth): For each $m \geq 1$, $\exists!$ Gorenstein curve singularity with $g=1$ and m branches:

$m=1$: cusp 

$C = V(y^2 - x^3)$

$m=2$: tacnode 

$C = V(y(y-x^2))$

$m=3$: planar triple pt 

$m \geq 3$: union of m general lines in $\mathbb{A}^{m-1}$.

"Elliptic m-fold point."

<u>E.g</u>: CUSP ($n=1$).

$$\mathcal{O}_C = \frac{K[x,y]}{x^3-y^2} = K[t^2,t^3] \subseteq K[t] = \mathcal{O}_{\tilde{C}}$$

$\Rightarrow \delta = 1 \Rightarrow g = 1$.

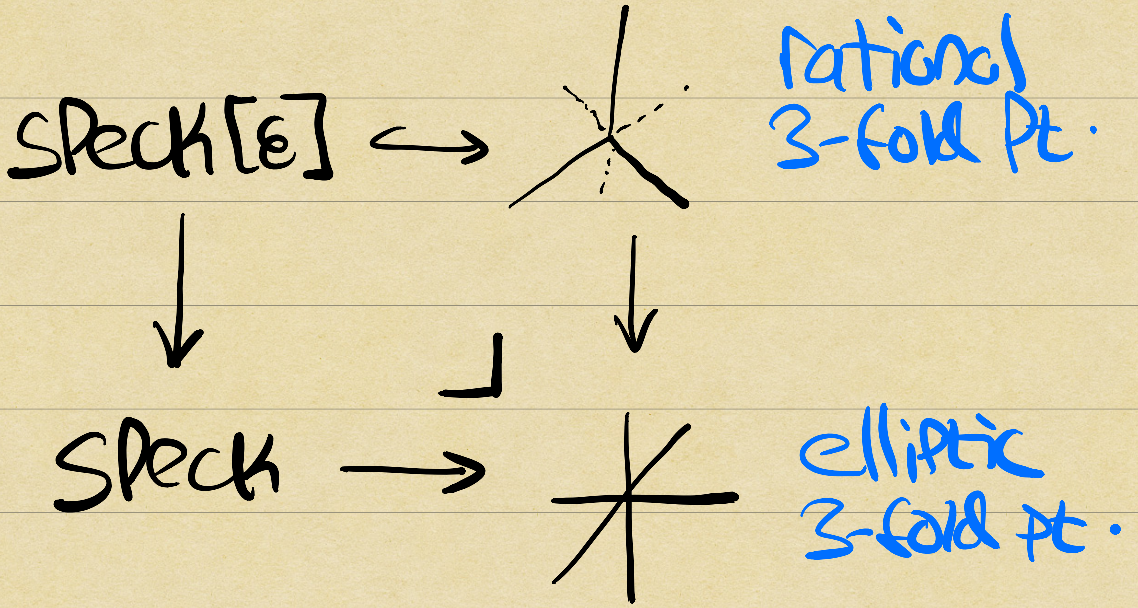
- $f(t) \in K[t] = \mathcal{O}_{\tilde{C}}$ descends to C iff $\partial f/\partial t|_{t=0} = 0$

E.g: Tacnode: glue 2 lines along a tangent vector.

$$
\begin{tikzcd}
\operatorname{Spec} k[\varepsilon] \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near end] & \mathbb{A}^1_s \arrow[d] \arrow[ddr, bend left] & \\
\mathbb{A}^1_t \arrow[r] \arrow[drr, bend right] & C = \underline{U} \arrow[dr, dashed] & \\
& & Z
\end{tikzcd}
$$

we see extra condition to descend!

E.g: Planar triple point: obtained from spatial triple point by collapsing general tangent vector.



- This is general: an elliptic m-fold pt is obtained from a rational m-fold pt by collapsing a general tangent vector.

  So see the 1 extra condition.

- Smyth uses these to give alternative modular compactifications of $M_{1,n}$.

- Viscardi ('10): Same for stable maps.

  Idea:
  1. disallow contracted elliptic components.
  2. allow elliptic m-fold singularities.

- Kills excess components!

- Battistella-Carocci-Manolache ('18): for the quintic 3-fold, Viscardi invariants agree with Li-Zinger reduced invariants (main component contribution).

- But Viscardi's space is not a desingularisation of the main component:

  1) It's not smooth in general. (It's relatively unobstructed, but space of smyth curves is singular.)

  2) There isn't even a map: $Vis_{1,n}(\mathbb{P}^r, d) \to \overline{M}_{1,n}(\mathbb{P}^r, d)$.

- Both Viscardi's space and the main component of stable maps contain the same dense open.

But there isn't a map in either direction.

To understand why, need to understand how Smyth curves are constructed.

(This will give connection to log structures.)

- **Q**: How to build a Smyth curve (C, P) from its pointed normalisation $(\tilde{C}, P_1 + \dots + P_m)$?

- **A**: Need the extra data of the hyperplane:

  $$\nu^* \Omega_{C,P} \subseteq \bigoplus_{i=1}^{m} \Omega_{\tilde{C}, P_i}$$

  3 ways to think of this:

  (i) consists of those functions on $\tilde{C}$ which descend to C.

  (ii) normal vector specifies the tangent vector to collapse.

(iii) identifies tangent spaces

$$T_{\tilde{C}, p_i} = T_{\tilde{C}, p_j} \quad (=T)$$

For $\tilde{f} : \tilde{C} \to Z$, gives:

$$\sum_i d\tilde{f}|_{p_i} : T \to T_{Z, \tilde{f}(q)}$$

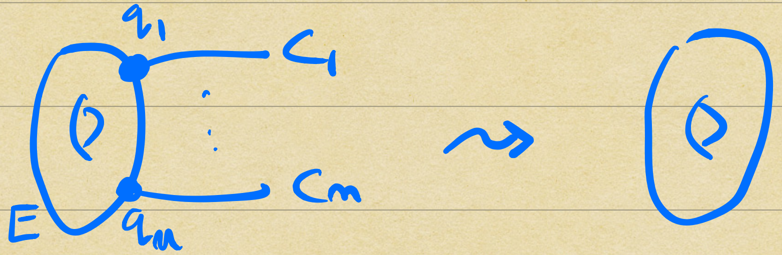
whose vanishing is equivalent to collapse of tangent vector in (ii).

"Moduli of attachments"

- Consequence: no way to go

Need more data!

• What if we smooth?



Get vector in normal space to stratum:

$$\bigoplus_{i=1}^{m} \left(T_{q_i} E \otimes T_{q_i} C_i \right).$$

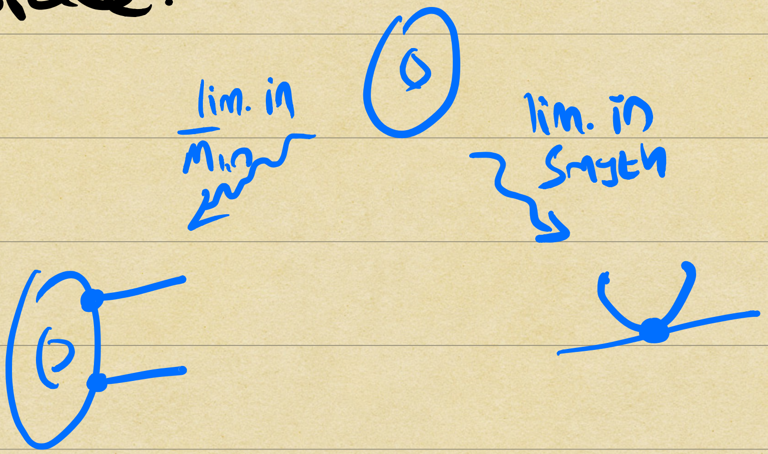
But $T_{q_i} E = T_{q_j} E$ (group law).
So really get vector in

$$\bigoplus_{i=1}^{m} T_{q_i} C_i$$

I.e. hyperplane in

$$\bigoplus_{i=1}^{m} \Omega_{q_i} C_i$$

Associated elliptic singularity is central fibre in Smyth space:



- Idea: add extra data to stable maps, to <u>encode a smoothing</u>:

Log structures!

3) Radial alignments (RSPW)

- Big Picture: blowup $\overline{M}_{1,n}(\mathbb{P}^r,d)$

$$\widetilde{VZ}_{1,n}(\mathbb{P}^r,d) \to \overline{M}_{1,n}(\mathbb{P}^r,d)$$

so that exceptional directions contain smoothing information.

Get contraction of universal curve:

$$\mathcal{C} \to \overline{\mathcal{C}}$$

Smyth curve

Pass to closed substack where map to $\mathbb{P}^r$ factors through $\overline{\mathcal{C}}$:

$$\begin{tikzcd} C \arrow[r] \arrow[dr, "f"'] & \overline{C} \arrow[d, dashed, "\overline{f}"] \\ & \mathbb{P}^r \end{tikzcd}$$

$\Longrightarrow$ Obstruction theory given by $H^i(\overline{C}, \overline{f}^* T_{\mathbb{P}^r})$.

Have:

$$\begin{tikzcd} VZ_{1,n}(\mathbb{P}^r, d) \arrow[r, hook] \arrow[d] & \widetilde{VZ}_{1,n}(\mathbb{P}^r, d) \arrow[d] \\ \overline{M}^{main}_{1,n}(\mathbb{P}^r, d) \arrow[r, hook] & \overline{M}_{1,n}(\mathbb{P}^r, d) \end{tikzcd}$$

Then $VZ_{1,n}(\mathbb{P}^r,d)$ is a desingularisation of the main component.

$$\begin{tikzcd} & VZ_{1,n}(\mathbb{P}^r,d) \arrow[dl] \arrow[dr] & \\ Vis_{1,n}(\mathbb{P}^r,d) \arrow[rr, dashed, leftrightarrow, "\sim"] & & \overline{M}^{main}_{1,n}(\mathbb{P}^r,d) \end{tikzcd}$$

.

- $\overline{M}_{1,n}(\mathbb{P}^r,d)$ has a log structure coming from its virtual boundary
- Gives tropicalisation:

$$T_{1,n}(\mathbb{P}^r,d) = \mathrm{Trop}\,\overline{M}_{1,n}(\mathbb{P}^r,d).$$

- What you need to know about $T_{1,n}(\mathbb{P}^r, d)$:

1) Plays role of "fan" of $\overline{M}_{1,n}(\mathbb{P}^r, d)$.

   ↳ • Is an abstract cone complex.

   • orbit-cone correspondence:

$$\left\{\begin{matrix}\text{cones in}\\ T_{1,n}(\mathbb{P}^r,d)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{boundary}\\ \text{strata in}\\ \overline{M}_{1,n}(\mathbb{P}^r,d)\end{matrix}\right\}.$$

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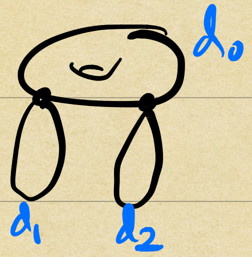
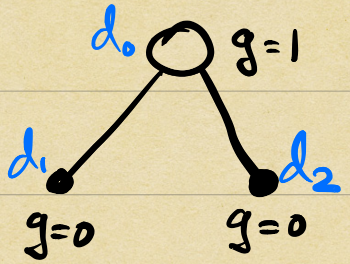
2) Can be thought of as a moduli space of tropical curves.

E.g.: Consider stratum in $\overline{M}_{1,n}(\mathbb{P}^r, d)$:



Associate dual graph:

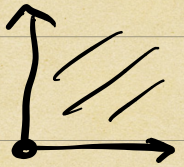


Then corresponding cone

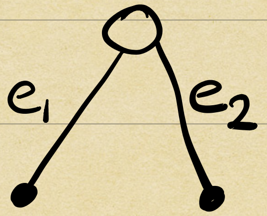
$$\sigma \in T_{1,n}(\mathbb{P}^r, d)$$

is:

$$\sigma \cong \mathbb{R}^2_{\geq 0} =$$



Think of as moduli for edge lengths:

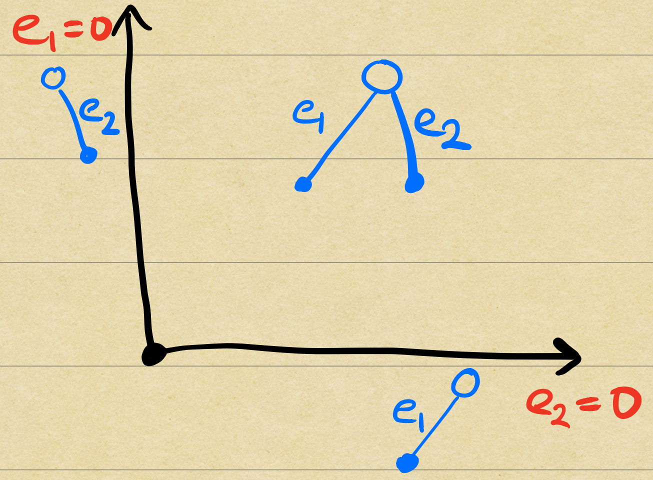


$$\sigma = (\mathbb{R}^2_{\geq 0})_{e_1, e_2}$$

Edge lengths encode "order of smoothing" of node. Normal directions to stratum:

$$\operatorname{codim} Z(\sigma) = \dim \sigma$$

Setting an edge length to 0 generises the curve, moving to a different stratum:



- Setting $e_i=0$ $\leftrightarrow$ Smoothing node q_i.

- face inclusions dual to strata inclusions.

3) Two key constructions from toric geometry carry over:

(i) Subdivision of $T_{1,n}(\mathbb{P}^r,d)$ $\rightsquigarrow$ birational modification of $\overline{M}_{1,n}(\mathbb{P}^r,d)$

(ii) PL function on $T_{1,n}(\mathbb{P}^r,d)$ $\rightsquigarrow$ Cartier divisor on $\overline{M}_{1,n}(\mathbb{P}^r,d)$.

- We'll use (i) to build $\widetilde{VZ}_{1,n}(\mathbb{P}^r,d) \to \overline{M}_{1,n}(\mathbb{P}^r,d)$.

- Idea: define another tropical moduli space, with map
$$\widetilde{T}_{1,n}(\mathbb{P}^r, d) \to T_{1,n}(\mathbb{P}^r, d).$$
which is a subdivision.

- Let's make precise statement that $T_{1,n}(\mathbb{P}^r, d)$ is moduli space of tropical curves.

- Fix τ a cone. Then a family of tropical curves over τ consists of:

(i) a graph Γ (w/ genus, degree, marking labels).

(ii) a map $\boxed{E(\Gamma) \xrightarrow{\varphi} \tau^\vee}$

- Picture: have family of curves:

$$\tilde{\Sigma} \xrightarrow{\pi} \tau$$

given $p \in \tau$, $\varphi(e)(p) \in \mathbb{R}_{\geq 0}$ is edge length of $e \in \pi^{-1}(p)$.

- This defines functor:

$$(\text{cones}) \xrightarrow{F} (\text{sets})$$

$$\tau \mapsto \left\{ \begin{array}{l} \text{families of} \\ \text{tropical} \\ \text{curves over } \tau \end{array} \right\} \Big/ \sim$$

And we have:

$$\circledast \quad F(-) = \mathrm{Hom}_{\text{cones}}(-, T_{1,n}(\mathbb{P}^r, d))$$

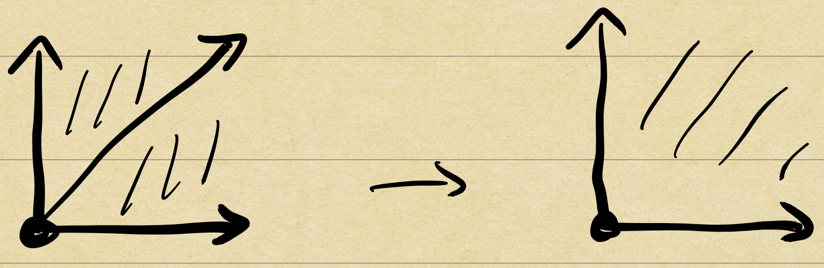
(Really should be a CFG; cf. Cavalieri–Chan–Ulirsch–Wise.)

- observation: if $\tilde{\Sigma} \to \Sigma$ is a subdivision, then

$$\mathrm{Hom}_{\text{cones}}(-, \tilde{\Sigma}) \subseteq \mathrm{Hom}_{\text{cones}}(-, \Sigma)$$

subfunctor.

<u>E.g.</u>:



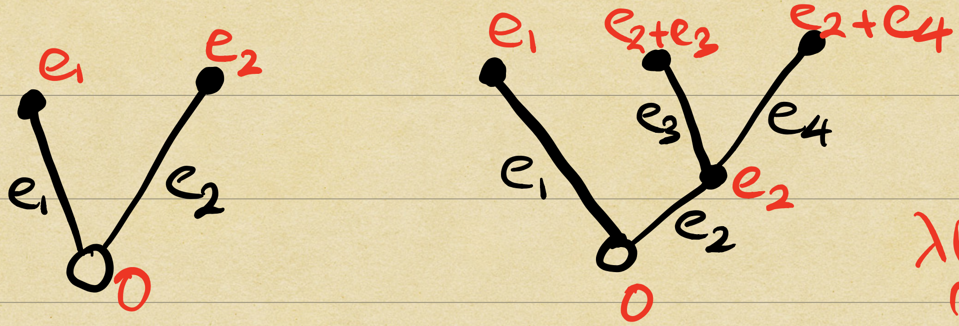
$\tilde{\Sigma}$ $\qquad\qquad$ Σ

To construct Smyth curve, want to identify tangent spaces at nodes adjacent to $g=1$ subcurve to be contracted.

- **Defn**: Γ a tropical curve over τ. For $v \in V(\Gamma)$, let

$$\lambda(v) \in \tau^{\vee} \quad (*)$$

be the sum of edge lengths connecting v to minimal $g=1$ subcurve ("core").



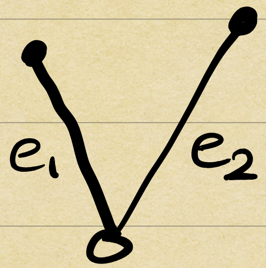
$\lambda(v)$ in red.

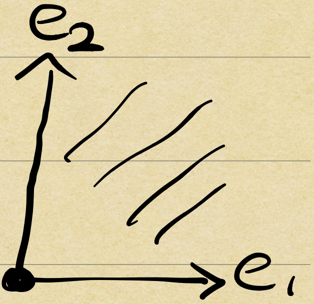
- Then Γ is radially aligned iff the $\lambda(v)$ are totally ordered.

$$\lambda(v) \geq \lambda(w) \iff \lambda(v) - \lambda(w) \in \tau^\vee.$$

- Let $\tilde{T}_{1,n}(\mathbb{P}^r, d)$ be moduli space of radially aligned curves. Clearly subfunctor of $T_{1,n}(\mathbb{P}^r, d)$.

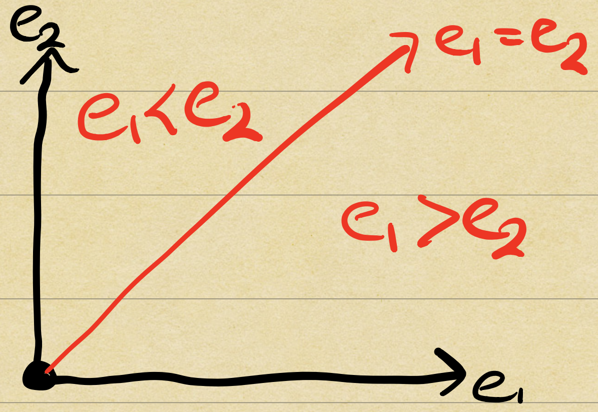
- How is this a subdivision of $T_{1,n}(\mathbb{P}^r, d)$?

E.g.: 

$\longleftrightarrow \sigma =$ 

Not radially aligned: e_1 and e_2 <u>not comparable</u> on σ.

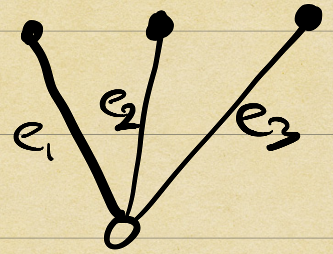
- Subdivide σ into regions:



Now comparable! This describes

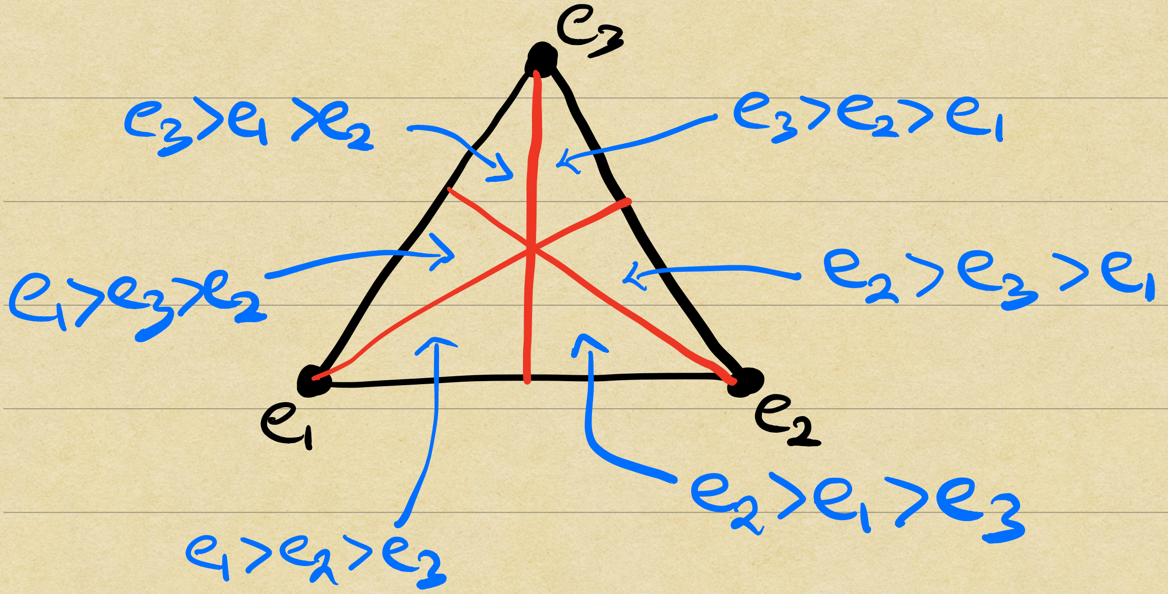
$$\widetilde{T}_{1,n}(\mathbb{P}^r, d) \to T_{1,n}(\mathbb{P}^r, d)$$

over the cone σ.

E.g.: 

, $\sigma \cong \mathbb{R}^3_{\geq 0}$

Height-1 slice of subdivision:



- Defines "birational" modification

$$\widetilde{VZ}_{1,n}(\mathbb{P}^r, d) \longrightarrow \overline{M}_{1,n}(\mathbb{P}^r, d).$$

- ⚠ Not really birational, b/c some strata of $\overline{M}_{1,n}(\mathbb{P}^r,d)$ have wrong codim.

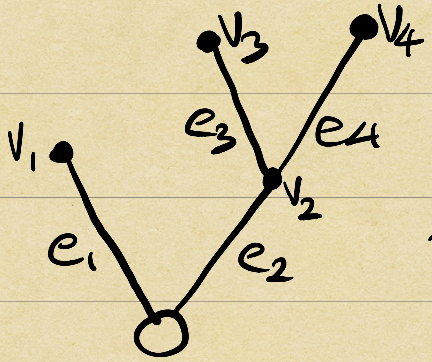
- Better to think of as pullback:

$$\begin{tikzcd}
\widetilde{VZ}_{1,n}(\mathbb{P}^r,d) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{1,n}(\mathbb{P}^r,d) \arrow[d] \\
\widetilde{\mathfrak{M}}^{wt}_{1,n} \arrow[r] & \mathfrak{M}^{wt}_{1,n}
\end{tikzcd}$$

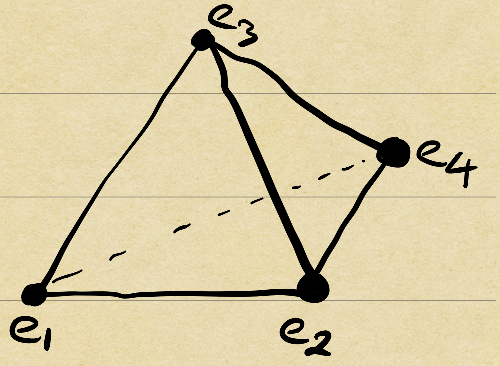
Related to, but different from, Hu-Li blowup.

- How to describe as an <u>iterated blowup</u>?

Pure combinatorics.



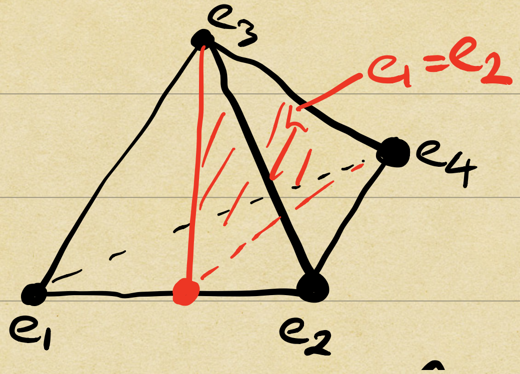
$\longleftrightarrow$



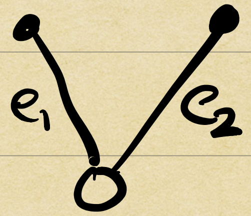
σ

- First step: Specify which $\lambda(v)$ is <u>minimal</u>.

2 candidates: $\left.\begin{array}{l}\lambda(v_1) = e_1 \\ \lambda(v_2) = e_2\end{array}\right\}$.



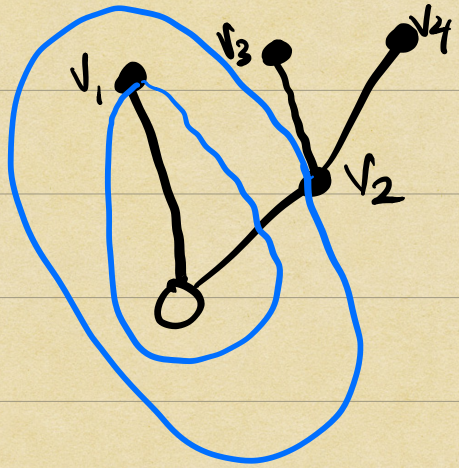
corresponds to blowup of stratum:



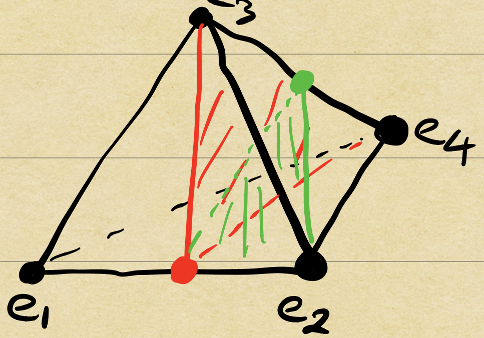
$(e_3 = e_4 = 0)$

- second step: decide which $\lambda(v)$ is next smallest.

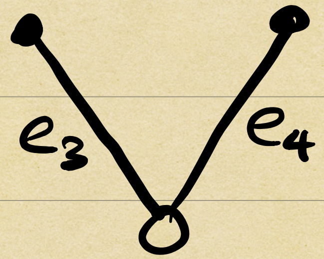
  Eg: on cone $\lambda(v_1) < \lambda(v_2)$, must have $\lambda(v_2)$ next smallest



- continue in this way. Next have to order e_3 and e_4:

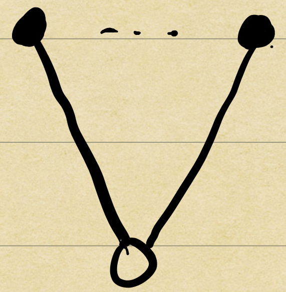


Amounts to blowing up stratum:



$(e_1 = e_2 = 0)$

- At each step, blowup stratum of form:



Stability ensures process respects a certain partial ordering.

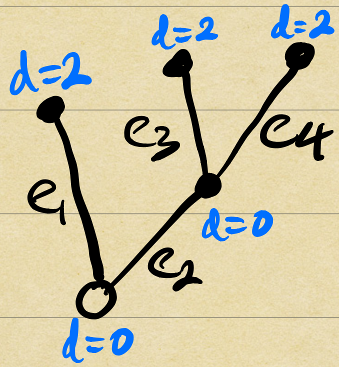
- Upshot:

$$\overline{VZ}_{1,n}(\mathbb{P}^r, d) \to \overline{M}_{1,n}(\mathbb{P}^r, d)$$

explicit iterated blowup.

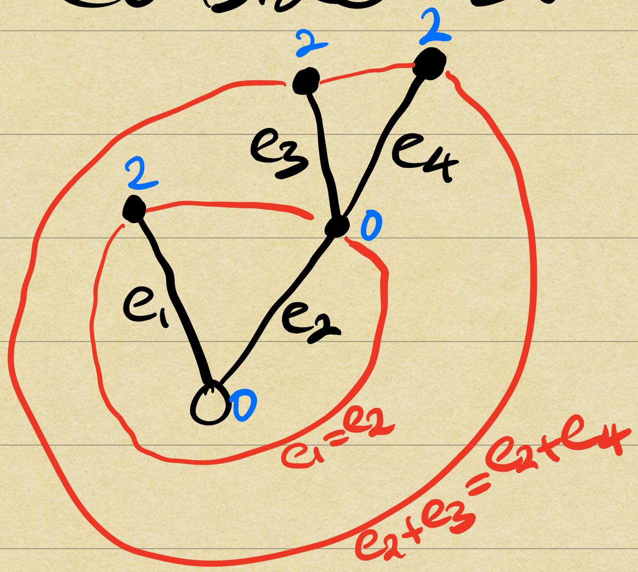
- Now connect back to Smyth curves story.

• E.g: Consider the stratum



in $\overline{M}_{1,n}(\mathbb{P}^r,d)$.

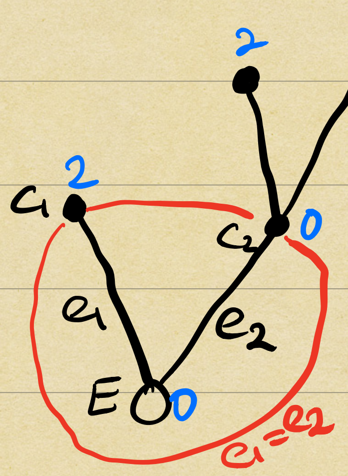
Consider stratum



lying over in $\sqrt{\mathcal{Z}}_{1,n}(\mathbb{P}^r,d)$.

• Has 2 fibre dimensions: identifications of tangent spaces

- Let δ = minimum $\lambda(v)$ where v is a vertex of deg > 0.
- Here $\delta = e_1 = e_2$.
- δ is PL function on $\widetilde{T}_{1,n}(\mathbb{P}^r, d)$, so defines line bundle $\mathcal{O}(\delta)$.



- $$\mathcal{O}(\delta) = T_{q_1} C_1 \otimes T_{q_1} E$$
$$= T_{q_2} C_2 \otimes T_{q_2} E.$$

- But $T_{q_1}E = T_{q_2}E$, so get identification:

$$T_{q_1}C_1 = T_{q_2}C_2$$

as promised.

- Produces contraction to a Smyth curve:

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[dr] & \overline{\mathcal{C}} \arrow[d] \\ & \widetilde{VZ}_{1,n}(\mathbb{P}^r,d) \end{tikzcd}$$

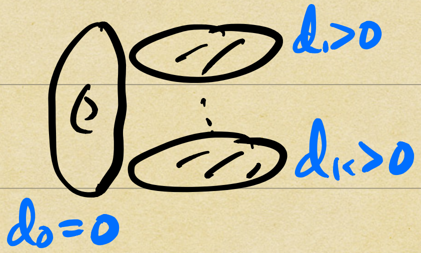
- Define $VZ_{1,n}(\mathbb{P}^r,d) \subseteq \widetilde{VZ}_{1,n}(\mathbb{P}^r,d)$ as locus where f factors:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d] & \mathbb{P}^r \\ \overline{\mathcal{C}} \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

(f unique if exists.)

- $VZ_{1,n}(\mathbb{P}^r,d) \to \widetilde{M}^{wt}_{1,n}$ is smooth: $H^1(E, f^*\mathcal{O}_{\mathbb{P}^r}(1))=0$.

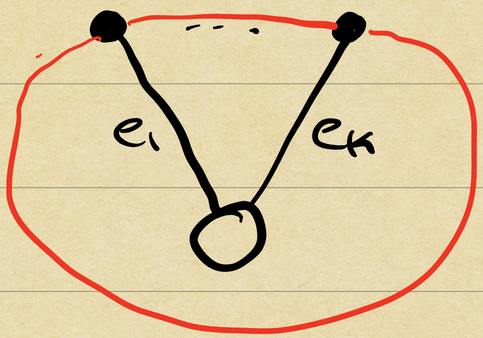
- what happened to excess cpts? Had:



Excess dim $= r-k$

in $\overline{M}_{1,n}(\mathbb{P}^r,d)$.

- Preimage in $\widetilde{VZ}_{1,n}(\mathbb{P}^r,d)$ has maximal stratum:



$e_1 = \dots = e_k$

Gain $k-1$ dims.

$\Rightarrow$ excess dim $= r-1$.

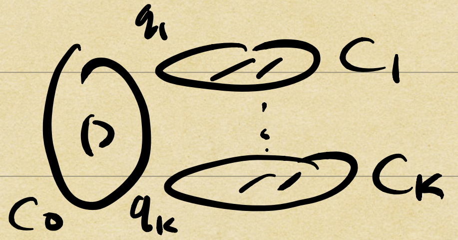
- Now impose <u>factorisation</u>: equivalent to vanishing of:

$$\sum_{i=1}^{k} df_i : \mathcal{O}(S) \to ev^* T_{\mathbb{P}^r}$$

cuts down by r dims.

$\Rightarrow$ end up w/ locus of codim. 1 in $VZ_{1,n}(\mathbb{P}^r, d)$.

- Vakil's criterion: in $\overline{M}_{1,n}(\mathbb{P}^r,d)$, a map of the form:



is in main component only if (iff?):

$$df_1(T_{C_1,q_1}),\dots,df_k(T_{C_k,q_k}) \in T_{\mathbb{P}^r,f(C_0)}$$

is linearly dependent.

- Factorisation says that a specific linear dependence holds (the one specified by the identifications

$T_{c_i, q_i} = T_{c_j, q_j}$).