Towards resolution of moduli of stable maps, II (joint w/ Yi Hu)

Tuesday, November 17, 2020 9:00 AM

- **Recall (from Yi's talk):**

  In order to resolve $\overline{\mathcal{M}}_g(\mathbb{P}^n,d) = \{ u: C \to \mathbb{P}^n \mid g(C)=g,\ \deg u = d,\ |\mathrm{Aut}(u)|<\infty \}$, we hope

  (I) to express the $\underbrace{\text{local equations of } \overline{\mathcal{M}}_g(\mathbb{P}^n,d)}$ as explicit as needed. given by the kernel of a $g\times d$ matrix M_φ } see Yi's slides

  (II) to globalize (I)

- To achieve Part (II) for $g=1,2$, we work on the relative Picard stack:

$$\mathfrak{P}_g = \{ (C,L) \mid g(C)=g,\ L\to C \text{ line bundle} \}$$

known: $\mathfrak{P}_g$ is a smooth Artin stack.

$$\overline{\mathcal{M}}_g(\mathbb{P}^n,d) \longrightarrow \mathfrak{P}_g, \qquad (C,u) \longmapsto (C,\ u^*\mathcal{O}_{\mathbb{P}^n}(1))$$

> Remark:
> for $g=1$, may use
> $\overline{\mathfrak{M}}_1^{wt} = \{ (C,w) \mid g(C)=1,\ w\in H^2(C,\mathbb{Z}),\ w(\Sigma)\geq 0\ \forall \text{ irr. } \Sigma\subset C \}$
> instead of $\mathfrak{P}_1$.

Goal:

to construct $\widetilde{\mathfrak{P}}_g \to \mathfrak{P}_g$ s.t. $\overline{\mathcal{M}}_g(\mathbb{P}^n,d)\times_{\mathfrak{P}_g}\widetilde{\mathfrak{P}}_g$ resolves $\overline{\mathcal{M}}_g(\mathbb{P}^n,d)$.

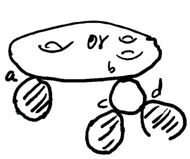
- To construct $\widetilde{\mathfrak{P}}_g$, notice that $\mathfrak{P}_g = \bigsqcup \underbrace{\mathfrak{P}_{(\gamma,\, w:\mathrm{Ver}(\gamma)\to\mathbb{Z})}}_{\{(C,L)\ \mid\ \text{dual graph of } C=\gamma,\ \deg L|_{C_v} = w(v)\}}$

Each $\mathfrak{P}_{(\gamma,w)}$ determines a rooted tree $\tau_{(\gamma,w)}$ s.t.

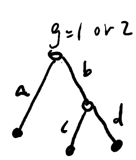
the root $\longleftrightarrow$ smallest genus g subcurve

the leaves $\longleftrightarrow$ the components w/ $\deg L>0$, "closest" to core.

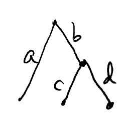
Ex.1



γ:

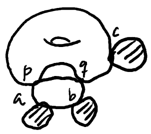


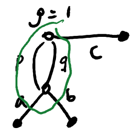
τ: same

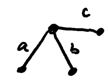


 : $w=\deg L|_{\dots} = 0$

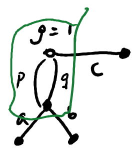
 : $w=\deg L|_{\dots} > 0$

Ex. 2. 

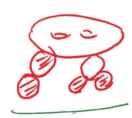
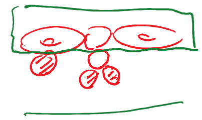
γ: 

τ: 

Ex. 3. 

γ: 

τ:

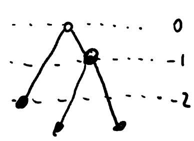
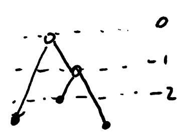
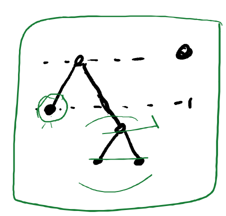
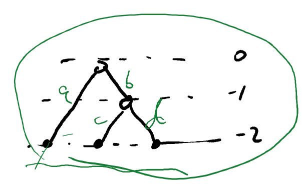
- Let $P_{[\tau]} = \bigsqcup_{\tau_{(\gamma,w)} = \tau} P_{(\gamma, w)}$ (e.g. ,  $\in P_{[\wedge]}$)

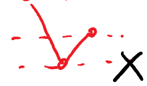
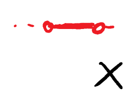
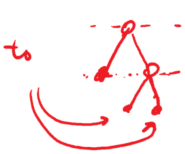
Then $P_g = \bigsqcup_{\text{all tree } \tau} P_{[\tau]}$

$P_{[\tau]}$ is smooth, locally given by $(\,\underset{\sim}{\rho_e} = 0 : e \in Edg(\tau))$

local parameters on $\mathcal{U} \to P_g$ corresp. to nodes.

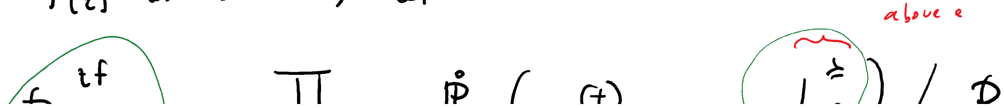
- After analyzing how VZ blowups affect each stratum $P_{[\tau]}$, we assign to each τ a "level structure" ℓ as follows:

e.g.     etc

Some rules:
- root always on top. 
- cannot have  
- actual values "0, -1, -2, ⋯" do <u>not</u> matter
- do not assign values to  due to VZ blowups

- For $P_{[\tau]}$ as above, let

$L_e \otimes \bigotimes_{\substack{\text{all } e' \\ \text{above } e}} L_{e'}$



$$\mathcal{P}^{tf}_{[\tau,\ell]} = \prod_{\substack{i \in \{\text{the} \\ \text{levels of } \ell\}}} \mathring{\mathbb{P}}\Big(\bigoplus_{\substack{\text{all edges } e \in Edg(\tau) \\ \text{s.t. the lower vertex of } e \\ \text{is on level } i}} \tilde{L}_e^{\succeq} \Big) \Big/ \mathcal{P}_{[\tau]}$$

above e

$\mathring{\mathbb{P}}(\oplus L_j)$
$= \{[v_j] \in \mathbb{P}(\oplus L_j) \mid v_j \neq 0 \ \forall j\}$

"twisted fields"

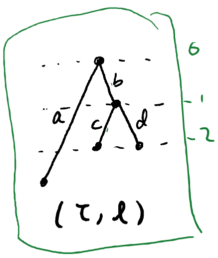
Remark: $L_e \to \mathcal{P}_{[\tau]}$ the line bundle s.t.

$L_e\big|_{\mathcal{P}_{[\tau]} \cap V}$

$=$ (the normal bdle of $(\varsigma_e = 0)$)$\big| \ldots$

$\mathcal{P}_{[\tau]} \cap V = (\varsigma_e = 0 : e \in Edg(\tau))$

e.g.



then $\mathcal{P}^{tf}_{[\tau,\ell]} = \underbrace{\mathbb{P}(L_b)}_{\substack{\text{level} \\ -1}} \times_{\mathcal{P}_{[\tau]}} \underbrace{\mathring{\mathbb{P}}(L_b \otimes L_c \oplus L_b \otimes L_d)}_{\substack{\text{level} \\ -2}}$

- Set theoretically, $\mathcal{P}^{tf}_g = \coprod_{\text{all } [\tau,\ell]} \mathcal{P}^{tf}_{[\tau,\ell]} \quad (\longrightarrow \mathcal{P}_g)$

In particular, over the main stratum $\mathcal{P}_{[\tau = \bullet]}$, no twisted fields added:

i.e. $\mathcal{P}^{tf}_{[\bullet]} = \mathcal{P}_{[\bullet]}$

- How to glue $\mathcal{P}^{tf}_{[\tau,\ell]}$ together to form $\mathcal{P}^{tf}_g$?

Answer: using smooth charts.

Ex. 4

Given $\tilde{x} \in \mathcal{P}^{tf}_{[\cdots]}$ over $x = (C, L) \in \mathcal{P}_{[\cdots]}$, let $V \longrightarrow \mathcal{P}_g$ be an affine smooth chart containing x w/. local parameters $\underbrace{\varsigma_a, \varsigma_b, \varsigma_c, \varsigma_d}_{\substack{\text{node smoothing,} \\ \text{vanish @ } x}}, \underbrace{\alpha_j}_{\text{other parameters}}$

We construct an affine smooth chart $\mathcal{U}_{\tilde{x}} \longrightarrow \mathcal{P}^{tf}_g$ w/. local parameters

$$\boxed{\underbrace{\xi_{-1}, \xi_{-2}, \xi_a}_{\text{vanishes @ } \tilde{x}}, \underbrace{\xi_d}_{\substack{\text{nowhere vanishing} \\ \text{on } \mathcal{U}_{\tilde{x}}}},} \ \alpha_j$$

by setting

by setting

- $\{\xi_{1} = \xi_{-2} = \xi_a = 0\} \longrightarrow P^{tf}_{[\;\cdots\;]}$

$$(0,0,0,\xi_d;\alpha_j) \longmapsto \Big(\underbrace{(0,0,0,0;\alpha_j)}_{\text{in } U \to P_{[\cdots]}},\ \Big[0 : 1\cdot\frac{\partial}{\partial\varsigma_b}\Big|_{(0,0,0,0;\alpha_j)}\Big],\ \Big[0 : 1\cdot\frac{\partial}{\partial\varsigma_b}\otimes\frac{\partial}{\partial\varsigma_c}\Big|_{\cdots} : \xi_d\,\frac{\partial}{\partial\varsigma_b}\otimes\frac{\partial}{\partial\varsigma_d}\Big|_{\cdots}\Big] \Big)$$

twisted fields

- $\{\xi_{-1} = \xi_{-2} = 0,\ \xi_a \neq 0\} \longrightarrow P^{tf}_{[\;\cdots\;]}$

$$(0,0,\xi_a,\xi_d;\alpha_j) \longmapsto \Big((0,0,0,0;\alpha_j),\ \Big[0 : \frac{\partial}{\partial\varsigma_b}\Big|_{\cdots}\Big],\ \Big[\xi_a\frac{\partial}{\partial\xi_a}\Big|_{\cdots} : \frac{\partial}{\partial\varsigma_b}\otimes\frac{\partial}{\partial\varsigma_c}\Big|_{\cdots} : \xi_d\,\frac{\partial}{\partial\varsigma_b}\otimes\frac{\partial}{\partial\varsigma_d}\Big|_{\cdots}\Big] \Big)$$

- $\{\xi_{-1} = 0,\ \xi_{-2} \neq 0,\ \xi_a = 0\} \longrightarrow P^{tf}_{[\;\cdots\;]}$

$$(0,\xi_{-2},0,\xi_d;\alpha_j) \longmapsto \Big((\underset{\varsigma_a}{0},\ \underset{\varsigma_b}{0},\ \underset{\varsigma_c}{\xi_{-2}},\ \underset{\varsigma_d}{\xi_{-2}\xi_d}),\ \Big[0 : \frac{\partial}{\partial\varsigma_b}\Big|_{\cdots}\Big] \Big)$$

- $\{\xi_{-1} = 0,\ \xi_{-2}, \xi_a \neq 0\} \longrightarrow P^{tf}_{[\;\cdots\;]}$

$$(0,\xi_{-2},\xi_a,\xi_d;\alpha_j) \longmapsto \Big((\underset{\varsigma_a}{0},\ \underset{\varsigma_b}{0},\ \underset{\varsigma_c}{\xi_{-2}},\ \underset{\varsigma_d}{\xi_{-2}\xi_d}),\ \Big[\xi_{-2}\xi_a\frac{\partial}{\partial\varsigma_a}\Big|_{\cdots} : \frac{\partial}{\partial\varsigma_b}\Big|_{\cdots}\Big] \Big)$$

- $\{\xi_{-1} \neq 0,\ \xi_{-2} = \xi_a = 0\} \longrightarrow P^{tf}_{[\;\cdots\;]}$

$$(\xi_{-1},0,0,\xi_d;\alpha_j) \longmapsto \Big((0,\xi_{-1},0,0),\ \Big[0 : \frac{\partial}{\partial\varsigma_c}\Big|_{\cdots} : \xi_d\frac{\partial}{\partial\varsigma_d}\Big|_{\cdots}\Big] \Big)$$

- $\{\xi_{-1} \neq 0,\ \xi_{-2} = 0,\ \xi_a \neq 0\} \longrightarrow P^{tf}_{[\;\cdots\;]}$

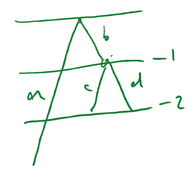
$$(\xi_{-1}, 0, \xi_a, \xi_d; \alpha_j) \longmapsto \left((0, \xi_{-1}, 0, 0), \left[\xi_{-1}\xi_a \frac{\partial}{\partial \varsigma_a}\Big|_{\dots} : \frac{\partial}{\partial \varsigma_c}\Big|_{\dots} : \xi_d \frac{\partial}{\partial \varsigma_d}\Big|_{\dots}\right]\right)$$

- $\{\xi_{-1} \neq 0, \xi_{-2} \neq 0, \xi_a = 0\} \longrightarrow P^{tf}_{[\cdot]} (= P_{[\cdot]})$

$$(\xi_{-1}, \xi_{-2}, 0, \xi_d; \alpha_j) \longmapsto (0, \xi_{-1}, \xi_{-2}, \xi_{-2}\xi_d)$$

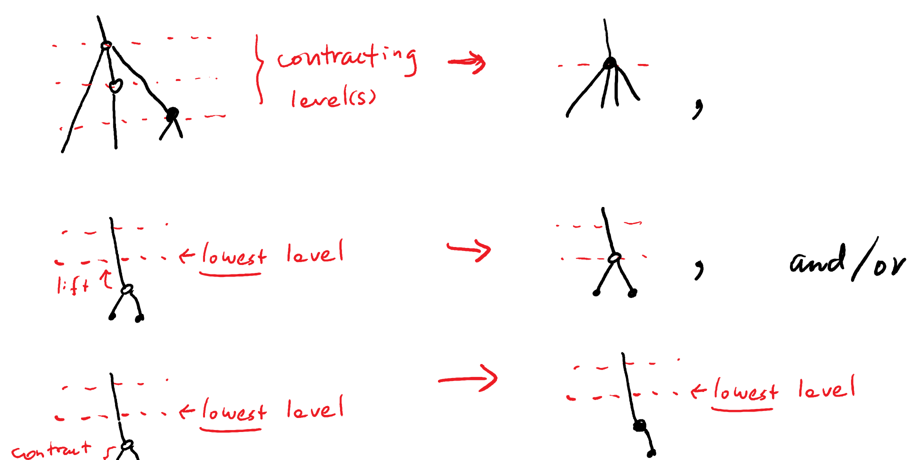
- $\overbrace{\{\xi_{-1} \neq 0, \xi_{-2} \neq 0, \xi_a \neq 0\}}^{\text{open in } \mathcal{U}_{\tilde{x}}} \longrightarrow P^{tf}_{[\cdot\cdot]} (= P_{[\cdot]})$

$$(\xi_{-1}, \xi_{-2}, \xi_a, \xi_d; \alpha_j) \longmapsto (\underbrace{\xi_{-1}\xi_{-2}\xi_a}_{\varsigma_a}, \underbrace{\xi_{-1}}_{\varsigma_b}, \underbrace{\xi_{-2}}_{\varsigma_c}, \underbrace{\xi_{-2}\xi_d}_{\varsigma_d}; \alpha_j)$$



Remark:

1. Intuitively, if $\tilde{x} \in P^{tf}_{[\tau,\ell]}$, then other possible $[\tau', \ell']$ in $\mathcal{U}_{\tilde{x}}$ are obtained from $[\tau, \ell]$ by



e.g. $\mathcal{M}^{tf}$ cannot be near $\mathcal{M}^{tf}$

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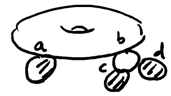
2. In the above example, the projection $\mathcal{U}_{\tilde{x}} \longrightarrow \mathcal{V}$ is given by

$$(\xi_{-1}, \xi_{-2}, \xi_a, \xi_d ; \alpha_j) \longmapsto (\underbrace{\xi_{-1}\xi_{-2}\xi_a}_{\rho_a}, \underbrace{\xi_{-1}}_{\rho_b}, \underbrace{\xi_{-2}}_{\rho_c}, \underbrace{\xi_{-2}\xi_d}_{\rho_d} ; \alpha_j)$$

3. For $g=1$, near ,

the matrix $M_\varphi = [\rho_a, \rho_b\rho_c, \rho_b\rho_d, \cdots]$ that locally defines $\overline{M}_1(\mathbb{P}^n, d)$

last time: $\rho_a w_a^j + \rho_b \rho_c w_c^j + \rho_b \rho_d w_d^j = 0$

$1 \le j \le n$

pulls back to

pull back

near

$$\longrightarrow \quad [\xi_{-1}\xi_{-2}\xi_a, \ \xi_{-1}\xi_{-2}, \ \xi_{-1}\xi_{-2}\xi_d, \ \cdots] \quad ✓$$

$$\underbrace{\xi_{-1}}\cdot\underbrace{\xi_{-2}} \cdot \left(\underbrace{w_c^j + \xi_a w_a^j + \xi_d w_d^j} \right) = 0$$

boundaries ↑ main component

Smooth

4. For $g=2$, near ,

(a, b not conjugate/Weierstrass)

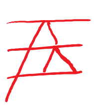
the matrix

each is a multiple of one of the entries on the left.

$$M_\varphi = \begin{bmatrix} \rho_b\rho_c & \rho_b\rho_c & \rho_b\rho_d & \rho_a & \cdots \\ 0 & \rho_b^2\rho_c^2 & \rho_b^2\rho_d & \rho_a & \cdots \end{bmatrix}$$

pulls back to

near



$$\longrightarrow \quad \xi_{-1}\xi_{-2}\begin{bmatrix} 1 & * & * & * & \cdots \\ 0 & \xi_{-1}\xi_{-2} & \xi_{-1} & \xi_a & \cdots \end{bmatrix} \quad ✓$$

Row 1 is in the desired form, but Row 2 is not yet!

Theorem 1. (Hu, -'19, '20)

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Let $\mathfrak{P}_g^{tf} = \bigsqcup \mathfrak{P}^{tf}_{[\tau, \ell]}$ be as above. Then, it glues to a smooth stack with the above $\mathcal{U}_{\tilde{x}}$'s as smooth charts.

- $\mathfrak{P}_g^{tf} \longrightarrow \mathfrak{P}_g$ is birational & proper.
- Locally, the products $\prod \mathfrak{s}_e$ pull back to desired forms

$\Rightarrow \overline{M}_1(\mathbb{P}^n, d) \times_{\mathfrak{P}_1} \mathfrak{P}_1^{tf}$ has smooth irreducible components and at worst normal crossing singularities.

- To deal with $g = 2$; we further notice

1). Generally, one can construct $\mathcal{M}^{tf} \longrightarrow \mathcal{M}$ for a (smooth) stack $\mathcal{M}$ as long as $\mathcal{M} = \bigsqcup_{\alpha \in I} \mathcal{M}_\alpha$ s.t.

a) each $\alpha \in I$ corresponds to a graph γ_α and an open cover $\{ \mathcal{V}_\alpha^i \xrightarrow{\mathfrak{s}_e} \mathcal{M} \}$ of $\mathcal{M}_\alpha$ by charts s.t.

s.t. $\mathcal{M}_\beta \cap \mathcal{V}_\alpha^i \neq \phi \Rightarrow$ γ_β is obtained from γ_α by edge contraction, $\mathcal{M}_\beta \cap \mathcal{V}_\alpha^i$ can be described by $\begin{cases} \mathfrak{s}_e \neq 0 \iff e \text{ contracted} \\ \mathfrak{s}_e = 0 \iff e \text{ not contracted} \end{cases}$

b) each $\alpha \in I$ corresponds to a rooted tree τ_α s.t. the edge contraction on τ_α is "compatible" w/ the edge contraction on γ_α.

Remark: we use γ_α's to control the local geometry of $\mathcal{M}$ and use τ_α's to add levels.

2). $\mathcal{M}^{tf}$ naturally has a stratification. There are choices of assigning γ_α's and τ_α's to $\mathcal{M}^{tf}$. This suggests a recursive construction

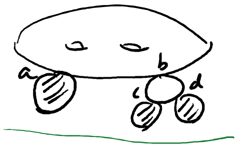
$$\cdots \longrightarrow (\mathcal{M}^{tf})^{tf} \longrightarrow \mathcal{M}^{tf} \longrightarrow \mathcal{M}$$

Theorem 2. (Hu, - '20)

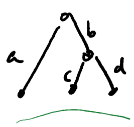
We can apply an 8-step construction to obtain $\underbrace{\widetilde{\mathbb{P}}_2^{+f} \longrightarrow \mathbb{P}_2}_{\text{smooth, proper, birat'l}}$ s.t.

$\overline{\mathcal{M}}_2(\mathbb{P}^n, d) \times_{\mathbb{P}_2} \widetilde{\mathbb{P}}_2^{+f}$ has smooth irreducible components and at worst normal crossing singularities.

Ex 4 (g=2 continued):

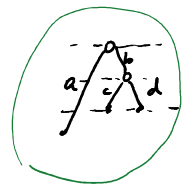


(nodes in general positions)



$\gamma = \tau$

$\xrightarrow{+\text{level}}$



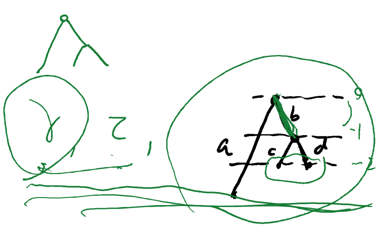
each is a multiple of one of the entries on the left.

$$M_\varphi = \begin{bmatrix} \varsigma_b \varsigma_c & \varsigma_b \varsigma_c & \varsigma_b \varsigma_d & \varsigma_a & \boxed{\cdots} \\ 0 & \varsigma_b^2 \varsigma_c^2 & \varsigma_b^2 \varsigma_d & \varsigma_a & \boxed{\cdots} \end{bmatrix}$$

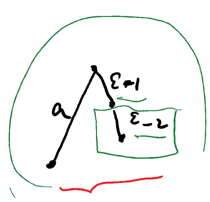
As aforementioned, M_φ pulls back to

$$\xi_{-1}\xi_{-2} \begin{bmatrix} 1 & * & * & * & \cdots \\ 0 & \xi_{-1}\xi_{-2} & \xi_{-1} & \xi_a & \cdots \end{bmatrix}$$

The 2^{nd} row is still not smooth — we keep adding "twisted fields":

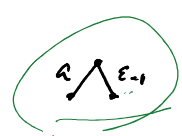


$\longrightarrow$



"derived graph" new γ_α

,

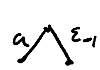


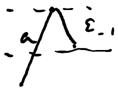
new tree τ_α

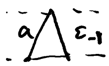
(warning: $\exists$ other choices of τ_α !)

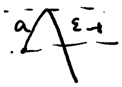
Proposition: they satisfy a) & b) above

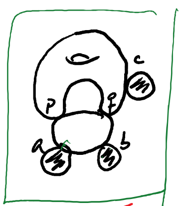
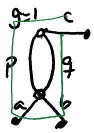
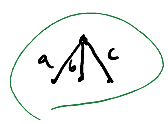
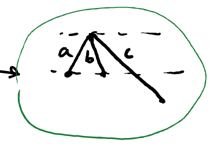
$\Rightarrow (\mathbb{P}_2^{+f})^{+f} \longrightarrow \mathbb{P}_2^{+f}$

There are only 3 ways of adding levels to :

 $\Rightarrow \begin{pmatrix}\hat\xi_1\xi_{-2} & 0\\ 0 & \xi_{-1}^{-2}\xi_{-2}\end{pmatrix}\begin{pmatrix}1 & * & \cdots\\ 0 & 1 & \cdots\end{pmatrix}$

 $\Rightarrow$ same form

 $\Rightarrow \begin{pmatrix}\hat\xi_1\xi_{-2} & 0\\ 0 & \xi_{-1}^{-2}\xi_{-2}\end{pmatrix}\begin{pmatrix}1 & * & \cdots\\ 0 & 1 & \cdots\end{pmatrix}$

Ex. 5    $\xrightarrow{+\text{ level}}$ 

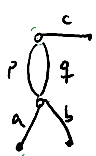
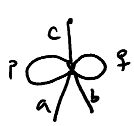
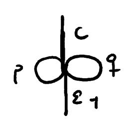
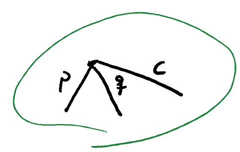
this forces the nodes a & b to be conjugate.

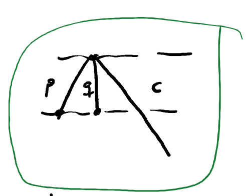
$$M_\varphi = \begin{pmatrix} \varsigma_a & g_a & \varsigma_b & \varsigma_c & \varsigma_c & \cdots \\ 0 & (f_a\varsigma_p + g_a\varsigma_q)\varsigma_a^2 & (\varsigma_p+\varsigma_q)\varsigma_b & (f_b\varsigma_p+g_b\varsigma_q)\varsigma_b^2 & \varsigma_c & \cdots\end{pmatrix},$$

where f's, g's, $\det\begin{pmatrix} f & 1\\ g & 1\end{pmatrix}$, $\det\begin{pmatrix} f_a & f_b\\ g_a & g_b\end{pmatrix} \in \Gamma(\mathcal{O}_\nu^*)$,

pulls back to

$$\xi_{-1}\begin{pmatrix}1 & * & * & * & * & \cdots\\ 0 & (f_a\varsigma_p+g_a\varsigma_q)\xi_a^2\xi_{-1} & \underbrace{\varsigma_p+\varsigma_q}_{=0 \iff a,b \text{ conjugate}} & (f_b\varsigma_p+g_b\varsigma_q)\xi_{-1} & \varsigma_c & \cdots\end{pmatrix}.$$

 $\longrightarrow$  $\xrightarrow{\text{derived graph}}$  + 

Consider near 

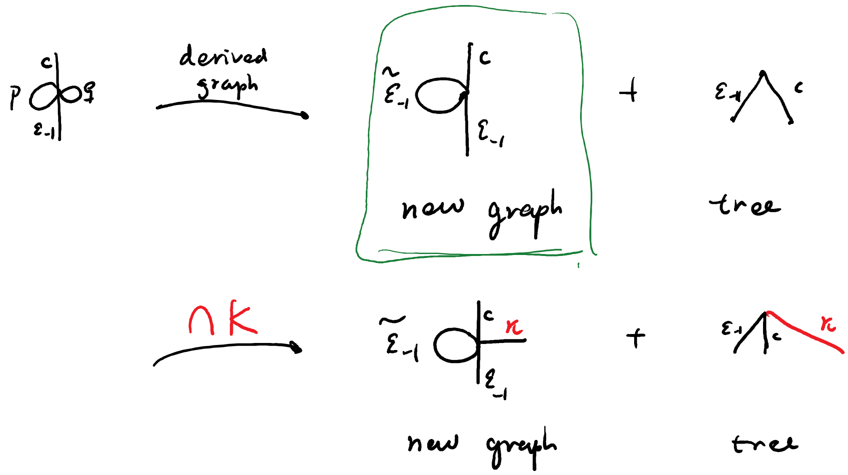
(for other level structures on  the pullback of M_φ is already diagonalized)

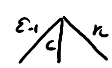
M_φ further pulls back to

M_φ further pulls back to

$$\begin{pmatrix} \cdots \end{pmatrix} \begin{pmatrix} 1 & * & * & * & \cdots \\ 0 & \underbrace{1+\xi_q}_{\text{a local parameter, denoted by } \kappa} & \xi_{-1} & \tilde{\xi}_c & \cdots \end{pmatrix}$$

- $\kappa = 0$ is global $\longleftrightarrow$ a, b conjugate
  denoted by K



Similar to Ex. 4, for any possible level added to ,
the pullback of M_φ is diagonalized.

Thank you !