Log twisted differentials.

joint with Dawei Chen.

1. Log twisted differentials.

Definition.

- $C \to S$: a family of log curve.
  - $(\underline{C}, \{p_i\}) \to \underline{S}$: pre-stable curve.
  - $\mathcal{M}_C / C$, $\mathcal{M}_S / S$: log str. .....
- $\underline{\omega}_{\underline{C}/\underline{S}}$: the dualizing sheaf.

  $\cup$
- 0_ω : the zero section. $\rightsquigarrow \mathcal{M}_0$
- $\mathcal{M}_{W_{C/S}} = \mathcal{M}_C|_{\underline{\omega}_{\underline{C}/\underline{S}}} \oplus_{\mathcal{O}^*} \mathcal{M}_0$

no poles at markings.

$$\Rightarrow \quad W_{C/S} := (\underline{\omega}_{\underline{C}/\underline{S}}, \mathcal{M}_{W_{C/S}})$$

- $$\begin{tikzcd} W_{C/S} \arrow[d] \\ C \arrow[d] \arrow[u, bend right, "\eta"'] \\ S \end{tikzcd}$$

  η : log twisted differential

  a special case of log maps.

Example 1

$$\underline{S} = \operatorname{Spec} \mathbb{C}, \quad \underline{C} : \text{smooth}.$$

Then $\eta \in H^0(\underline{\omega}_{\underline{C}/\underline{S}})$ s.t.

$$\eta^* \mathcal{O}_{\omega_{C/S}} = \underbrace{\sum \mu_i \cdot p_i}_{\text{forced by log.}}$$

markings.

$$\Rightarrow \deg \underline{\omega}_{\underline{C}/\underline{S}} = 2g-2 = \sum \mu_i$$

μ_i : the contact order at the i-th marking.

It is constant over a connected family of log twisted differentials.

$$\overline{\mathcal{H}}(\mu_1, \cdots, \mu_n) := \left\{ (C/S, \eta) \;\middle|\; \begin{array}{l} \eta \text{ has contact order} \\ \mu_i \text{ at } p_i \end{array} \right\}$$

Thm. $\overline{\mathcal{H}}(\mu_1, \cdots, \mu_n)$ is a separated log DM-stack.

Its log structure is called the minimal/basic log structure.

How to think about η ?

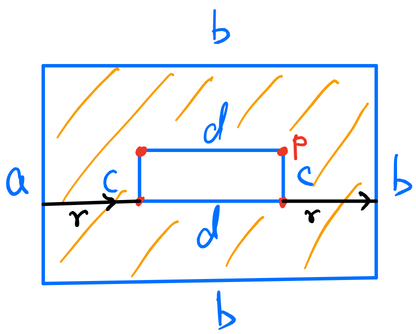
$$\begin{array}{c} (\underline{C}, \eta) \\ \downarrow \\ S = (\operatorname{Spec} \mathbb{C}, M_S) \end{array}$$

- normalization : $\bigsqcup_i \underline{Z}_i \longrightarrow \underline{C}$
- $\eta|_{\underline{Z}_i}$: meromorphic differential

$\mathbb{C}^*$

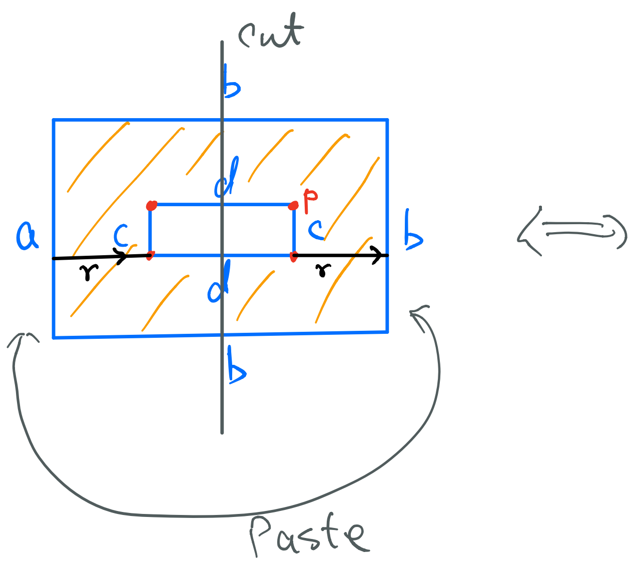
- Choose a chart :

$\overline{M}_S := M_S / \mathbb{C}^* \longrightarrow M_S \iff$ choose $\{\tilde{\eta}|_{Z_i}\}_i$ in

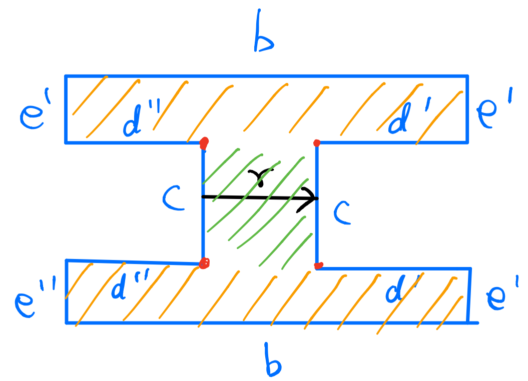
Ex. 

$\in \mathcal{H}(2).$

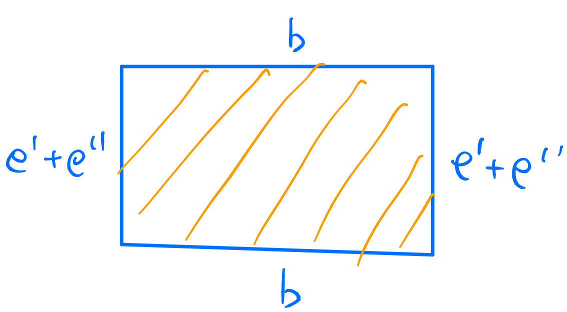
Q : $r \longrightarrow 0$?



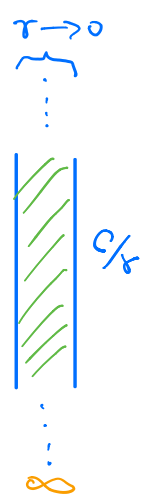
$\Longleftrightarrow$

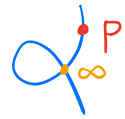


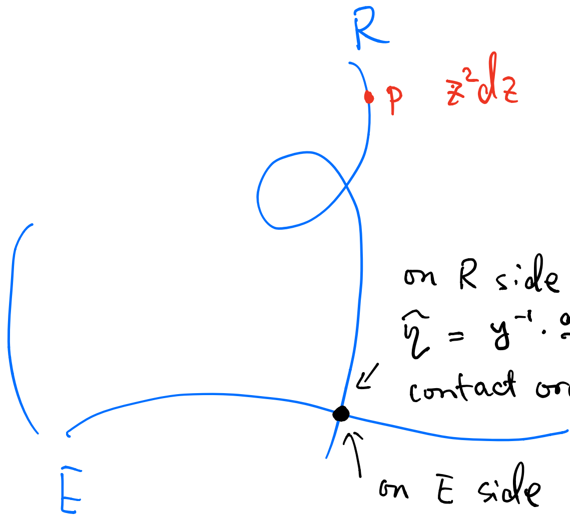
Let $r \longrightarrow 0$:



$+$



Elliptic curve $+$ 

$=$ (

on R side

$\hat{\eta} = y^{-1} \cdot \frac{dy}{y}$

contact order $= -1$

on E side

$\eta = z \cdot \frac{dz}{z}$

contact order $= 1$

$+ \quad \eta$)

The spin structure.

- Suppose $(C, \eta) \in \mathcal{H}(\mu_1, \cdots, \mu_n)$. s.t. $2 \mid \mu_i$.
- 2-spin: $S := \mathcal{O}(\sum \frac{\mu_i}{2} p_i) \Rightarrow S^2 = \omega_C$.
- (C, η) has $\begin{cases} \text{even spin} & \text{if } h^0(S) \equiv 0 \bmod 2 \\ \text{odd spin} & \text{if } h^0(S) \equiv 1 \bmod 2 \end{cases}$.

Fact: $\mathcal{H}(\mu_1, \cdots, \mu_n) = \mathcal{H}^{odd}(\mu_1, \cdots, \mu_n) \cup \mathcal{H}^{even}(\mu_1, \cdots, \mu_n)$

Thm $\overline{\mathcal{H}}(\mu_1, \cdots, \mu_n) = \overline{\mathcal{H}}^{odd}(\mu_1, \cdots, \mu_n) \cup \overline{\mathcal{H}}^{even}(\mu_1, \cdots, \mu_n)$

How to define spin parity ?

$(C \longrightarrow S, \eta)$:

$$
\begin{tikzcd}
 & \eta^* M_0 \arrow[r, "\eta^b"] \arrow[d] & M_C \arrow[d] & S^* \arrow[l, hook'] \arrow[d] \arrow[l, phantom, "\square", yshift=-25pt] & S \arrow[l, phantom, "\supset"] & W \setminus O_W \arrow[d] \\
\mathbb{N} \arrow[r] & \overline{M}_0|_C \arrow[r, "\bar{\eta}^b"] & \overline{M}_C & \frac{\Theta}{2} \arrow[l, phantom, "\ni"] & & \theta \\
1 \arrow[rr, mapsto] & & \Theta & & &
\end{tikzcd}
$$

Then the spin parity is defined as the parity of $h^0(\zeta)$ as before.

But for $\frac{\Theta}{2} \in \overline{M}_C$, we need

① contact orders are even at both markings and nodes.

② for each general $z_i \in Z_i$, the degeneracy

$$\Theta|_{z_i} \in \overline{M}_C|_{z_i} \simeq \overline{M}_S$$

is divisible by 2.

For ①

$$\begin{tikzcd}
\tilde{C} \arrow[r] \arrow[d] & C \arrow[d] \\
\tilde{S} \arrow[r] & S
\end{tikzcd}$$

take $\sqrt[2]{}$ - stack at nodes with odd contact orders.

log orbifold curves.

finite choices of changing log structures.

For ②

$$\begin{tikzcd} \tilde{\mathcal{C}}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \tilde{\mathcal{C}} \arrow[d] \\ \tilde{S}' \arrow[r] & \tilde{S} \end{tikzcd}$$

Any refinement of $\overline{M}_{\tilde{S}}$ s.t. $2 | \theta|_{z_i}$ $\forall i$.

$\Longrightarrow$ 2-Spin : $\mathcal{S}^2 \simeq \omega_{\tilde{\mathcal{C}}'/\tilde{S}'}$

Prop - Define

suspose minimal.

$(\mathcal{C}/S, \eta)$ is $\begin{cases} \text{even if } H^0(\mathcal{S}) \equiv 0 \bmod 2 \\ \text{odd if } H^2(\mathcal{S}) \equiv 1 \bmod 2. \end{cases}$

Proof of thm. The construction $\mathcal{S}$ can apply to a connected family. Then it follows from Abramovich-Jarvis.

Proof of Prop-Def.

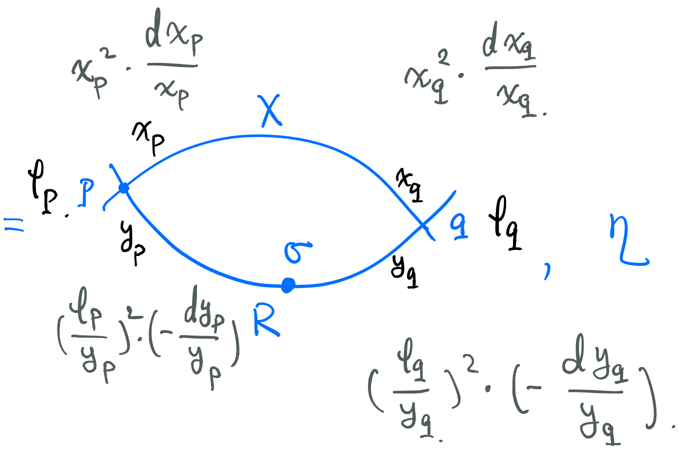
① $\mathcal{S}$ is representable.

② $\bigsqcup_i Z_i \longrightarrow \tilde{\Sigma}'$: normalization at orbi-fold nodes.

③ Show that $\xi|_{Z_i}$ are independent of choices.

Ex

$$\left(C = \right.$$



$$\left. , \ \eta \right).$$

C
$\downarrow$
S.

$$g(X) = 2, \quad g(R) = 3, \quad \mu_\sigma = 4$$

$$c_p = c_q = 2.$$

$$(\eta_X) = p + q,$$

$$(\eta_R) = 4\sigma - 3p - 3q.$$

$$\Rightarrow \begin{cases} \omega_{C/S}|_X = \omega_X(p+q) = \omega_X^{\otimes 2} \\ \omega_{C/S}|_R = \omega_R(p+q) \simeq \mathcal{O}_R. \end{cases}$$

$$\Rightarrow \begin{cases} \xi|_X = \omega_X \\ \xi|_X = 0 \end{cases} \qquad H^0(\omega_X) = 2$$

the minimal one.

Note $S = \mathrm{Spec}(\mathbb{N} \to k)$, $\overline{M}_S = \mathbb{N}$.

$\ell \in M_S$.

ℓ_p, ℓ_q: smoothing of p and q.

Two non-isomorphic η:

$$\begin{cases} \ell_p \mapsto \ell \\ \ell_q \mapsto \ell \end{cases} \quad \text{or} \quad \begin{cases} \ell_p \longmapsto \ell \\ \ell_q \longmapsto -\ell. \end{cases}$$

One smooth to $\underbrace{\text{even spin}}_{\text{coincide with Hyp. in } \mathcal{H}(4)}$, one smooth to odd spin.

This is when section glues.

Hyper-elliptic loci

$$\mathrm{Hyp} = \left\{ \left(\underbrace{\begin{tikzcd} C \arrow[rr] \arrow[dr] & & P \arrow[dl] \\ & S & \end{tikzcd}}_{\text{log admissible cover}} , \underbrace{\eta}_{\substack{\text{log twisted} \\ \text{diff. over } C}} \right) \;\middle|\; \iota^* \eta = -\eta \right\}$$

$$\Big\downarrow \text{ log sm}$$

$$\text{log sm} \quad \mathcal{Q} = \left\{ \left(\begin{matrix} P \\ \downarrow \\ S \end{matrix} , \underbrace{q}_{\substack{\text{corresponding} \\ \text{quadratic differential.}}} \right) \right\}$$

$\Longrightarrow$ Hyp is log smooth.