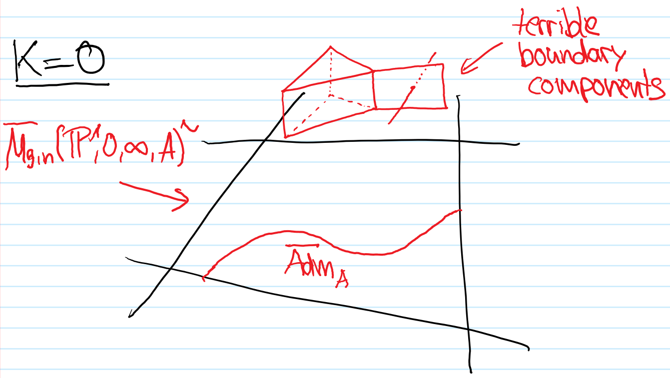
A spontaneous comment to Tom's talk

$K=0$

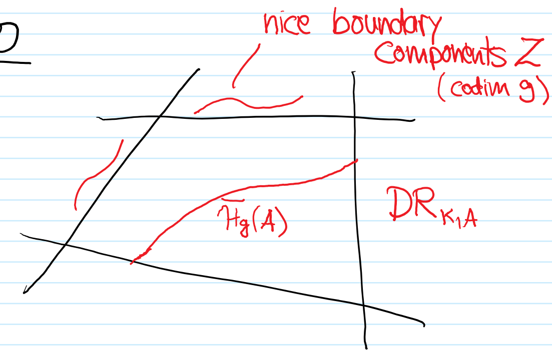


$$\begin{pmatrix}\text{explicit}\\ \text{formula}\end{pmatrix} \underset{\substack{\uparrow\\ [JPPZ16]}}{=} DR_{0,A} = [\overline{Adm}_A] + ???$$

$\underline{Q}$ Smooth space $\widetilde{M} \xrightarrow{\pi} \overline{M}_{g,n}(\mathbb{P}^1,0,\infty,A)^\sim$

s.t. $\pi_*[\widetilde{M}]^{vir} = [\ldots]^{vir}$

$K>0$



$$\begin{pmatrix}\text{explicit}\\ \text{formula}\end{pmatrix} \underset{\substack{\uparrow\\ [BHPSS20]}}{=} DR_{K,A} \underset{\substack{\uparrow\\ [HS19]}}{=} [\overline{\mathcal{H}}_g(A)] + \sum_Z m_Z \cdot [Z]$$

$\rightsquigarrow$ recursive computation of $[\overline{\mathcal{H}}_g(A)]$

Multiplicativity of the DR cycles in the bChow ring

Recall Given k, $A=(a_1,\dots,a_n)\in\mathbb{Z}^n$ w/ $\sum a_i = k(2g-2)$

We defined cycle $\overline{DR}_{k,A}\in CH^g(\overline{M}_{g,n})$ compactifying $\{(C,p_1,\dots,p_n) : \omega_C^{\otimes k}\cong\mathcal{O}_C(\sum a_ip_i)\}$

smooth

↑ suppress from notation

$$\begin{tikzcd}
M_{g,n}^A \arrow[r, "\tilde\sigma"] \arrow[d, hook, "\text{open immers.}"'] & \mathrm{Pic}^0 \arrow[d, bend left, "e=\mathcal{O}_C"'] \\
\overline{M}_{g,n}^A \arrow[r, "f"', "\text{log blowup}"] & \overline{M}_{g,n} \arrow[u, dashed, bend left, "\omega_C^{\otimes k}(-\sum a_ip_i)=\sigma"]
\end{tikzcd}$$

$\widetilde{DRL} = \tilde\sigma^{-1}(e)\subset M_{g,n}^A$

$\widetilde{DRC} = \tilde\sigma^*([e])$ cycle support. on $\widetilde{DRL}$

Thm $\widetilde{DRL}\xrightarrow{f}\overline{M}_{g,n}$ proper

Def $\overline{DR}_A := f_*\widetilde{DRC}\in CH^g(\overline{M}_{g,n})$

Note f is isomorphism over $M_{g,n}^{ct}$

⇝ $\sigma_A^{ct}: M_{g,n}^{ct}\longrightarrow \mathrm{Pic}^0$ extends (twist $\omega_C^{\otimes k}(-\sum a_ip_i)$ by vertical divisors)

$$\overline{DR}_A\Big|_{M_{g,n}^{ct}} = (\sigma_A^{ct})^*[e]$$

Small observation

$(C,p_1,\dots,p_n)\in M_{g,n}$

$$\begin{cases}\omega_C^{\otimes k}\cong\mathcal{O}_C(\sum a_ip_i)\\ \omega_C^{\otimes k'}\cong\mathcal{O}_C(\sum b_ip_i)\end{cases} \iff \begin{cases}\omega_C^{\otimes k}\cong\mathcal{O}_C(\sum a_ip_i)\\ \omega_C^{\otimes(k+k')}\cong\mathcal{O}_C(\sum(a_i+b_i)p_i)\end{cases}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$DRL_A\cap DR_B \quad = \quad DRL_A\cap DRL_{A+B} \subseteq M_{g,n}$$

Q What about intersection product of cycles?

Prop (HPS 17)

$$\overline{DR}_A\cdot\overline{DR}_B = \overline{DR}_A\cdot\overline{DR}_{A+B}\in CH^{2g}(M_{g,n}^{ct}) \qquad (☆)$$

but in general not in $CH^{2g}(\overline{M}_{g,n})$.

Proof (compact type statement)

$$\begin{tikzcd}
\mathrm{Pic}^0 \arrow[d] \\
M_{g,n}^{ct} \arrow[u, bend left, "\sigma_B"] \arrow[u, bend right, "\sigma_A"']
\end{tikzcd}$$

$\overline{DR}_A = \sigma_A^*[e]$ supported on $\{\sigma_A=e\}$ (*)

$$\begin{aligned}
\overline{DR}_A\cdot\overline{DR}_B &= \overline{DR}_A\cdot(\sigma_B\oplus e)^*[e]\\
&\overset{(*)}{=} \overline{DR}_A\cdot(\sigma_B\oplus\sigma_A)^*[e]\\
&= \overline{DR}_A\cdot\sigma_{A+B}^*[e]\\
&= \overline{DR}_A\cdot\overline{DR}_{A+B} \qquad \square
\end{aligned}$$

Rmk $\overline{DR}_A = \overline{DR}_{-A}\in CH^g(\overline{M}_{g,n})$ ⇝ $\omega_C^{\otimes k}(-\sum a_ip_i)\cong\mathcal{O}_C$

$\iff \omega_C^{\otimes -k}(-\sum(-a_i)p_i)\cong\mathcal{O}_C$

$\Leftrightarrow \omega_C^{\otimes -k}(-\sum(-a_i)p_i) \cong \mathcal{O}_C$

<u>Exa</u> $\overline{M}_{1,2}$, $k = k' = 0$

$$\begin{matrix} A = (3,-3) \\ B = (5,-5) \end{matrix} \rightsquigarrow \begin{matrix} (3,-3) \\ (2,-2) \end{matrix} \rightsquigarrow \begin{matrix} (1,-1) \\ (2,-2) \end{matrix} \rightsquigarrow \begin{matrix} (1,-1) \\ (0,0) \end{matrix}$$

(Euclidean algorithm)

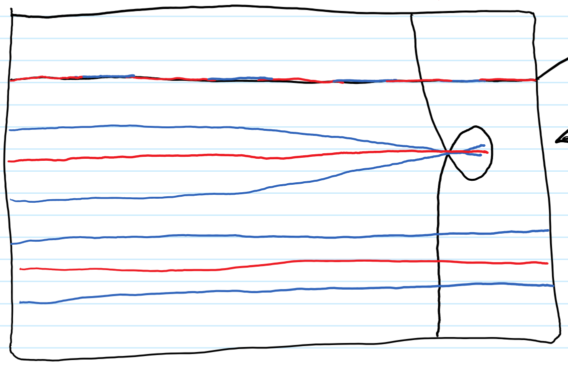
More generally:

$$\begin{matrix} A = (a,-a) \\ B = (b,-b) \end{matrix} \rightsquigarrow \begin{matrix} (\gcd(a,b), -\gcd(a,b)) =: GCD \\ (\quad 0 \quad , \quad 0 \quad) \end{matrix}$$

If ($\star$) were true in $CH^2(\overline{M}_{1,2}) = \mathbb{Q}\cdot[pt]$:

$$\underbrace{\overline{DR}_A \cdot \overline{DR}_B}_{\frac{1}{24}(a^2b^2 - a^2 - b^2)} \overset{!}{=} \underbrace{\overline{DR}_{GCD} \cdot \overline{DR}_{\underline{0}}}_{-\frac{1}{24}\gcd(a,b)^2}$$

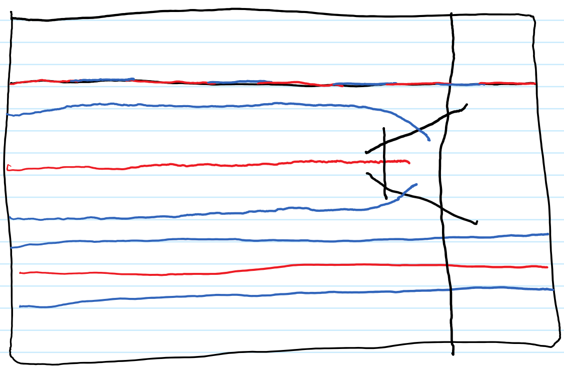
<u>Exa</u> $\gcd(a,b) = 1$ $\overline{DR}_A$ $\overline{DR}_B$



self-intersect. $-\frac{1}{24}$

lots of intersect. at 

$\uparrow$ blow-up resolving σ_A <u>and</u> σ_B



no more intersection

$\widetilde{DRC}_A \cdot \widetilde{DRC}_B = -\frac{1}{24}$

$\Rightarrow$ Instead of computing $\overline{DR}_A \cdot \overline{DR}_B \in CH^{2g}(\overline{M}_{g,n})$, should compute intersection in suitable blowup

$\rightsquigarrow$ bChow ring

<u>Def</u> S irreducible, noetherian DM stack.

$$bCH^*(S) = \operatorname*{colim}_{X \to S} CH^*(X)$$

$X \to S$ proper, birational, representable, X regular

For
$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[dr] & X \arrow[d] \\ & S \end{tikzcd}$$
: $[(X \to S, \alpha \in CH^*(X))]$
$= [(X' \to S, g^*\alpha \in CH^*(X'))]$

Filtered colimit

$$\begin{tikzcd} & \exists Z \arrow[dl, dashed] \arrow[dr, dashed] & \\ \alpha \quad X \arrow[dr] & & Y \quad \beta \arrow[dl] \\ & S & \end{tikzcd}$$

$\rightsquigarrow$ Compute $\alpha + \beta$, $\alpha \cdot \beta$ on Z

Note $\exists$ natural pushforward

$$bCH^*(S) \longrightarrow CH^*(S)$$
$$[(X \xrightarrow{f} S, \alpha)] \longmapsto f_*\alpha$$

DR cycles are naturally born in $bCH^*(\overline{M}_{g,n})$:

$$\begin{tikzcd} \widetilde{DRC}^X_A \quad DRL^X_A \arrow[r, hook] \arrow[r, bend left, "i"] & X_0 \subseteq X \arrow[r] & \overline{M}_{g,n} \end{tikzcd}$$

$\widetilde{DRC}^X_A = \sigma_A^*[e]$, $DRL^X_A = \sigma_A^{-1}(e)$

X_0: largest open on which $\sigma_A : X \dashrightarrow Pic^e$ extends

(**) i proper $\rightsquigarrow$ $\widetilde{DR}^X_A = i_* \widetilde{DRC}^X_A \in CH^g(X)$

Thm (Holmes)
The $(X \to \overline{M}_{g,n})$ with (**) form a cofinal system
and the net $[(X \to \overline{M}_{g,n}, \widetilde{DR}^X_A)] \in bCH^g(\overline{M}_{g,n})$
is eventually constant.

$\rightsquigarrow$ every $X_0 \to \overline{M}_{g,n}$ dominated by $X \to X_0 \to \overline{M}_{g,n}$ w/ (**)

Def Let $\widetilde{DR}_A \in bCH^g(\overline{M}_{g,n})$ be the limit of this net.

Note $\widetilde{DR}_A \mapsto \overline{DR}_A \in CH^g(\overline{M}_{g,n})$.

Thm (HPS)
$\widetilde{DR}_A \cdot \widetilde{DR}_B = \widetilde{DR}_A \cdot \widetilde{DR}_{A+B} \in bCH^{2g}(\overline{M}_{g,n})$.

Pf Choose X_A st. not is const. from $[(X_A \to \overline{M}_{g,n}, \widetilde{DR}_A^{X_A})]$,

" X_B " " " " " $[(X_B \to \overline{M}_{g,n}, \widetilde{DR}_B^{X_B})]$,

" X_{A+B} " " " " " $[(X_{A+B} \to \overline{M}_{g,n}, \widetilde{DR}_{A+B}^{X_{A+B}})]$.

$$\begin{tikzcd} & X \arrow[dl] \arrow[d] \arrow[dr] & \\ X_A \arrow[dr] & X_{A+B} \arrow[d] & X_B \arrow[dl] \\ & \overline{M}_{g,n} & \end{tikzcd}$$

Check $\widetilde{DRL}_A^X \cap \widetilde{DRL}_B^X = \widetilde{DRL}_A^X \cap \widetilde{DRL}_{A+B}^X$

and since $\widetilde{DR}_A^X = \sigma_A^*[e]$, ...

can repeat proof as on $M_{g,n}^{ct}$.

$\square$

Open question

double double ramification cycle

What is the image $DDR(A,B) \in CH^{2g}(\overline{M}_{g,n})$ of $\widetilde{DR}_A \cdot \widetilde{DR}_B$

under $bCH^{2g}(\overline{M}_{g,n}) \longrightarrow CH^{2g}(\overline{M}_{g,n})$?

Exa $g=1, n=2$

$\overline{M}_{1,2}$ rational $\Rightarrow \forall X \to \overline{M}_{1,2}$ rational $\Rightarrow CH^2(X) \cong \mathbb{Q}\cdot[pt]$

$$\Rightarrow \quad \begin{matrix} bCH^2(\overline{M}_{1,2}) & \xrightarrow{\sim} & CH^2(\overline{M}_{1,2}) \\ \| & & \| \\ \mathbb{Q} & & \mathbb{Q} \end{matrix}$$

Using Thm:

$$DDR\begin{pmatrix} a & -a \\ b & -b \end{pmatrix} = DDR\begin{pmatrix} \gcd(a,b) & -\gcd(a,b) \\ 0 & 0 \end{pmatrix}$$

Note $\sigma_0 = e$ always extends

$$\begin{tikzcd} X_{GCD} \arrow[d, "f"] \\ \overline{M}_{1,2} \end{tikzcd}$$

$$\begin{aligned} &DDR\begin{pmatrix} GCD \\ 0 \end{pmatrix} \\ &= f_*\left(\widetilde{DR}_{GCD}^{X_{GCD}} \cdot \widetilde{DR}_0^{X_{GCD}}\right) \\ &= f_*\left(\widetilde{DR}_{GCD}^{X_{GCD}} \cdot f^*(\overline{DR}_0)\right) \\ &= \overline{DR}_0 \cdot f_* \widetilde{DR}_{GCD}^{X_{GCD}} \\ &= \overline{DR}_0 \cdot \overline{DR}_{GCD} = -\frac{1}{24}\gcd(a,b)^2 \cdot [pt]. \end{aligned}$$

Variant Triple double ram. cycles in $g=1, n=3$

$$\begin{pmatrix} a & * & * \\ b & * & * \\ c & * & * \end{pmatrix} \rightsquigarrow \begin{pmatrix} \gcd & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \rightsquigarrow \begin{pmatrix} \gcd & * & * \\ 0 & \gcd' & * \\ 0 & 0 & 0 \end{pmatrix} \leftarrow \text{since row sums} = 0$$

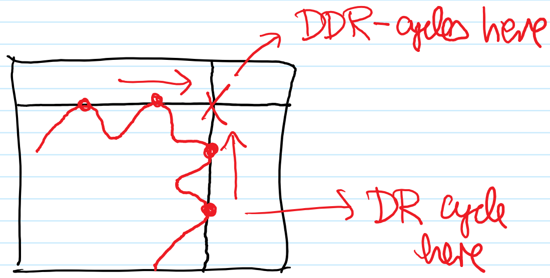
↳ $DR_{\underline{0}} = (-1)^g \cdot \lambda_g$, $\lambda_g|_{\overline{M}_{g,n} \setminus M_{g,n}^{ct}} = 0$

$\Rightarrow$ answer only depends on DR on $M_{g,n}^{ct}$

$\rightsquigarrow$ explicit formula (Buryak-Rossi)

Ongoing projects

▷ Graber, Pandharipande, Ranganathan, Zvonkine
   ↳ use localization to obtain formulas



▷ Holmes, Ranganathan, Schwarz
   ↳ develop language for / compute in bChow tautological ring of $\overline{M}_{g,n}$