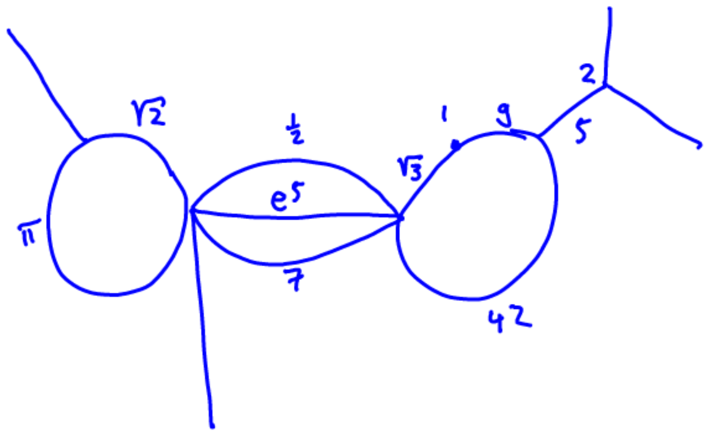
Artin fans & the moduli stack of tropical curves

(j.w. R. Cavalieri, M. Chan, J. Wise [CCUW20])

Def.: An (abstract) tropical curve consists of

- a vertex-weighted finite graph
  $G = (V, E, L, h: V \to \mathbb{Z}_{\geq 0})$
- an edge length $d: E \to \mathbb{R}_{>0}$



$g = 4 + 1 + 2 = 7$

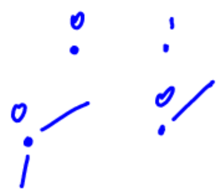
Def.: • The genus of G (resp. Γ) is

$$g(G) := g(\Gamma) := \underbrace{b_1(G)}_{\#E - \#V + 1} + \sum_{v \in V} h(v)$$

• G (resp. Γ) is stable if $\forall v \in V$

$$2h(v) - 2 + \underbrace{\mathrm{val}(v)}_{\substack{= \#\text{ of edges} \\ \text{emanating} \\ \text{from } v}} > 0$$

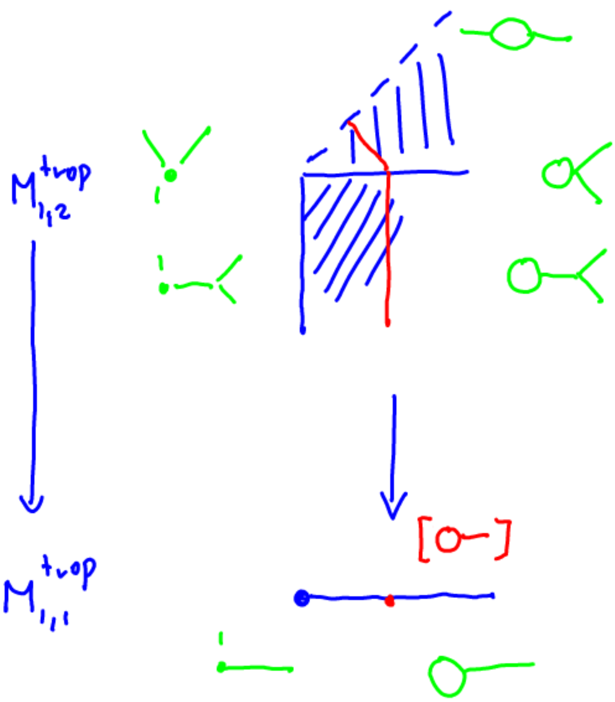
Not stable



Set-theoretic Def.:

$$M_{g,n}^{trop} := \left\{ [\Gamma] \;\middle|\; \begin{array}{l} \Gamma \text{ stable tropical curve of genus } g \\ \text{with } n \text{ ordered legs} \end{array} \right\}$$

E.g.:



Don't panic!

The big analogy

alg. geom.	trop. geom.
rings	monoids
affine scheme	rat'l polyhedral cones
schemes	cone complexes
DM-stacks	cone stacks

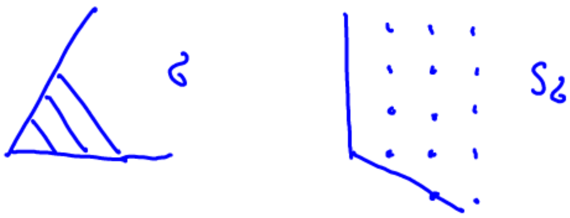
étale morphisms

strict morphisms

Def.: A *rat'l polyhedral cone* σ is a topological space $|\sigma|$ together with a f.g. free abelian subgroup $M \subseteq C^0(|\sigma|, \mathbb{R})$ s.t. the image of

$$\begin{aligned} |\sigma| &\longrightarrow \mathrm{Hom}(M, \mathbb{R}) =: N_{\mathbb{R}} \\ x &\longmapsto (m \mapsto m(x)) \end{aligned}$$

is an intersection finitely many integral half spaces



Fact: $S_\sigma := \{ m \in M \mid m(x) \geq 0 \}$ is a fin.gen. integral saturated sharp monoid

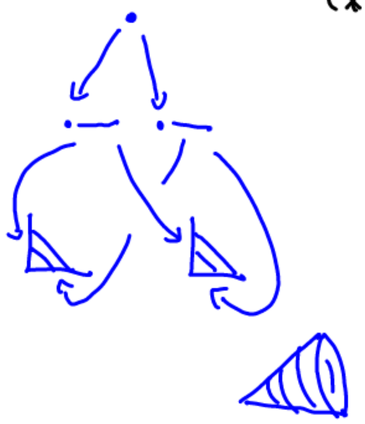
Def.:
- $\mathrm{Hom}(\sigma, \sigma') := \{ f : \sigma \to \sigma' \text{ continuous s.t. } f^*(m') \in M \ \forall m' \in M' \}$
- A *face* τ of σ is a subset of the form
$$\tau := \{ x \in \sigma \mid m(x) = 0 \}$$
for some $m \in M$

Fact: τ canonically is a rat'l poly. cone

- $f \in \mathrm{Hom}(\sigma, \sigma')$ is called a *face morphism* if f induces an isomorphism of σ onto a face of σ'

Def.: A <u>(rat'l. poly.) cone complex</u> is a category Σ fibered in sets over RPC^{face} s.t.

$$(*) \quad \# \mathrm{Hom}_{\Sigma}(\sigma, \sigma') \leq 1$$



The Waffle Cone is not a cone complex

Obs.: Compare this to presentation of a scheme by affine Zariski open subset !

Def.: $f \in \mathrm{Hom}_{RPCC}(\Sigma, \Sigma')$ is called <u>strict</u>, if $\forall \sigma \in \Sigma$ $f|_{\sigma} : \sigma \longrightarrow \Sigma'$ induces an isomorphism onto a cone σ' in Σ'

is strict

Fact: Strict surjective morphisms define a Grothendieck topology on RPCC

<u>Def.:</u> A <u>cone stack</u> $\mathcal{C}$ is a stack over $RPCC_{strict}$ s.t.

(i) $\Delta: \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is representable by cone complexes

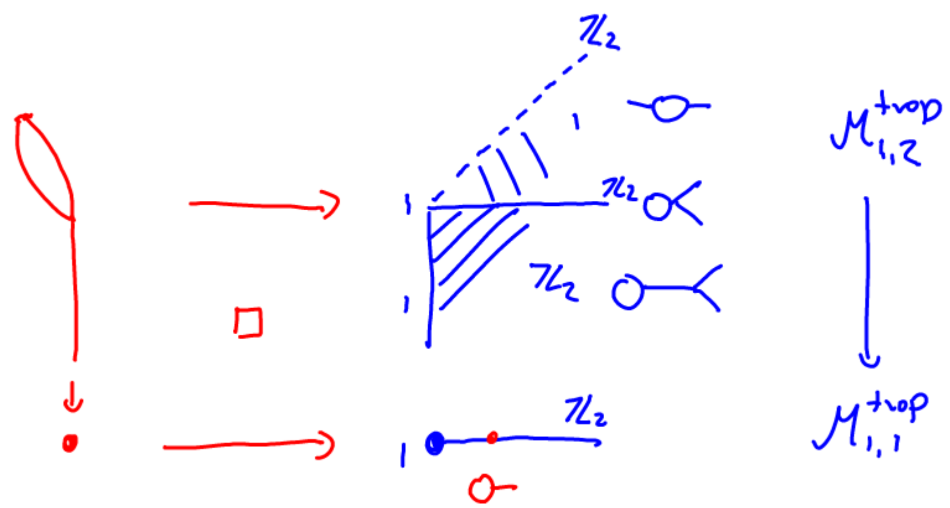
(ii) There is a strict surjective morphism
$$\Sigma \to \mathcal{C}$$
from a cone complex

<u>Prop./Def.:</u> There is a unique stack $\mathcal{M}_{g,n}^{trop}$ over $RPCC_{strict}$ s.t.

$$\mathcal{M}_{g,n}^{trop}(\sigma) = \left\{ \Gamma = (G, d: E(G) \to S_\sigma - \{0\}) \;\middle|\; \begin{array}{l} G \text{ stable of genus } g \\ \text{with } n \text{ marked legs} \end{array} \right\}$$

<u>Thm</u> (CCUW'20) $\mathcal{M}_{g,n}^{trop}$ is a cone stack

Now we have a universal curve ?

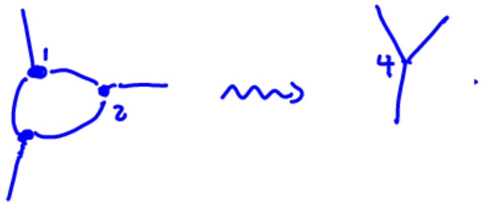


Sketch of proof:

$$\begin{array}{ccc}
\text{Cone stacks} & \xleftrightarrow{\;1-1\;} & \text{CFGs}/\mathrm{RPC}^{\text{face}} \\
\mathcal{C} & \longmapsto & \text{strict maps } G \mapsto \mathcal{C} \\
\mathrm{Hom}(-,\mathcal{D}) & \longleftarrow\!\!\shortmid & \mathcal{D}
\end{array}$$

Let $J_{g,n} :=$

- stabe weighted graphs of genus g with n marked legs
- weighted edge contractions



$$\begin{array}{cc}
G & J_{g,n} \\
\downarrow & \downarrow \\
\mathbb{R}_{\geq 0}^{E(G)} & \mathrm{RPC}^{\text{face}}
\end{array}$$

This a CFG & the associated cone stack is $\mathcal{M}_{g,n}^{trop}$ $\blacksquare$

Artin fans & tropicalization

Prop./Def. There is a full & faithful functor

$$a: \mathrm{RPC} \hookrightarrow \text{log.alg.stacks}$$
$$\sigma \longmapsto \left[\operatorname{Spec} k[S_\sigma] \big/ \underset{\operatorname{Spec} k[M]}{\underset{\shortparallel}{\mathbb{G}_m \otimes N}}\right] =: \mathcal{A}_\sigma$$

$\mathcal{A}_\sigma$ is called an Artin cone

$$\sigma = \mathbb{R}^2_{\geq 0} \rightsquigarrow S_\sigma = \mathbb{N}^2 \rightsquigarrow \mathcal{A}_\sigma = \left[\mathbb{A}^2 / \mathbb{G}_m^2\right]$$

$$\begin{tikzcd} \mathbb{G}_m \arrow[r] & \mathbb{G}_m^2 \\ 1 \arrow[u] \arrow[r] & \mathbb{G}_m \arrow[u] \end{tikzcd}$$

Def.: An Artin fan is a log.alg.stack $\mathcal{A}$ that admits a surjective strict étale cover from a disjoint union of Artin cones

Obs.: There is an equivalence

$$\text{Cone Stacks} \xrightarrow{\sim} \text{Artin fans}$$
$$\mathcal{C} \longmapsto a^* \mathcal{C}$$

Abuse of notation: Write $\mathcal{C}$ for $a^*\mathcal{C}$

⚠ There is another definition of Artin fans as log-alg.stacks log-étale / K. This is not equivalent to this one

Fact: $S = (\underline{S}, M_S)$ f.s. log. scheme

$$\mathcal{M}^{trop}_{g,n}(S) := \left\{ (\Gamma_s)_{\substack{s \to S \\ \text{geom. pt.}}} \;\middle|\; \begin{array}{c} \Gamma_s \in \mathcal{M}^{trop}_{g,n}(\overline{M}_{S,s}) \\ \text{compatible with} \\ \text{specialization} \end{array} \right\}$$

Thm (CCUW'20, U'19, ACP'15)

The morphism

$$\begin{aligned} \mathrm{trop}_{g,n} : \mathcal{M}^{log}_{g,n} &\longrightarrow \mathcal{M}^{trop}_{g,n} \\ (X/S) &\longmapsto (\Gamma_{X_s})_{s \hookrightarrow S} \end{aligned}$$

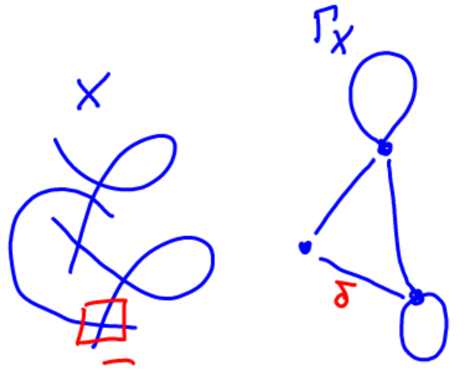
is well-defined, smooth, strict, surjective

and

$$\begin{tikzcd} \mathcal{M}^{log}_{g,n} \arrow[rr, "\text{strict}"] \arrow[dr] & & \mathcal{A} \\ & \mathcal{M}^{trop}_{g,n} \arrow[ur, dashed, "\exists!"] & \end{tikzcd}$$

$\mathcal{A}$ = Artin fan w/ faithful monodromy

i.e. $\mathcal{C}_{\mathcal{A}} \longrightarrow RPC$ is faithful



$\overline{M}_X = \mathbb{N}\alpha \oplus \mathbb{N}\beta /_{\alpha+\beta=\delta}$

$\delta \in \overline{M}_S$

Towards a logarithmic compactification of strata of abelian differentials

$$\mu = (m_1, \dots, m_n) \in \mathbb{Z}^n_{\geq 0} \quad \text{s.t.} \quad m_1 + \dots + m_n = 2g-2$$

Moduli of multi-scale differentials

$!\|?$

closure of $\mathcal{H}_g(\mu)$

[MUW'17]

$$\begin{tikzcd}
\mathcal{H}_g^{\log}(\mu) \subseteq \widetilde{\mathcal{M}}_{g,n}^{\log} \arrow[rr] \arrow[d, phantom, "\cup|"] \arrow[dd, bend left] & & \mathcal{H}_g^{\mathrm{trop,realizabe}}(\mu) \arrow[d, "\text{non-proper subdivision + Kummer map}"] \\
\mathcal{H}_g(\mu) \arrow[d] & \square & \mathcal{H}_g^{\mathrm{trop}}(\mu) \arrow[d, "\text{non-proper subdivision}"] \\
\mathcal{M}_{g,n} \subseteq \mathcal{M}_{g,n}^{\log} \arrow[rr, "\mathrm{trop}_{g,n}"] & & \mathcal{M}_{g,n}^{\mathrm{trop}}
\end{tikzcd}$$

$$\mathcal{H}_g^{\log}(\mu) =: \text{closure of } \mathcal{H}_g(\mu)$$

$$\left\{ (X, \vec{p}) \;\middle|\; \mathcal{O}_X\Big(\sum_{i=1}^n m_i p_i\Big) = \omega_X \right\} =: \mathcal{H}_g(\mu)$$

$$\mathcal{H}_g^{\mathrm{trop}}(\mu) := \left\{ \Gamma \in M_g^{\mathrm{trop}} \;\middle|\; \sum_{i=1}^n m_i r(l_i) \sim K_\Gamma \right\}$$