MAT952 Expander graphs (HS21)

Johannes Schmitt

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The main reference for the seminar will be the lecture notes [Kow] by Kowalski. Below are the outlines of the topics of individual talks and the concepts and results that should be explained. Depending on the amount of material in the sections below (which refer to [Kow] unless stated otherwise), it might be necessary to either shorten the presented material (e.g. by leaving out technical details in proofs) or to add more material (e.g. by discussing some exercise, working out some additional example or filling in details of arguments). In some cases there are suggestions for additional material.

1. Introduction to graphs (Sect. 2.1, p. 11–20)
   Definition of a graph (with examples and drawings), adjacency matrix, basic families of graphs (cycle, path, complete graph, example of Cayley graph), maps, isomorphisms and embeddings of graphs, girth, trees and forests
   Additional material: Show that a graph is a tree iff it is connected and its number of vertices is precisely one more than its number of edges. More examples of graphs, compute automorphism groups of some families of graphs.

2. Basic properties of graphs (Sect. 2.2, p. 20–29)
   Bipartite graphs, Paths and distance, diameter, universal cover (only sketch proof of Proposition 2.2.14), omit Proposition 2.2.17
   Additional material: Work out some of the exercises; show that for adjacency matrix A of graph G, the (i, j)-th entry of A^m is equal to the number of paths from i to j of length m

3. Cayley graphs, action graphs, Schreier graphs (Sect. 2.3, p. 29–39)
   Cayley graphs (with examples), metric properties, bipartiteness criterion,

automorphism group, action graphs, Schreier graphs
Additional material: More examples and nice pictures

4. Expansion in graphs (Sect. 3.1, p. 40–50)
   Expansion constant, example computations, bounds, expander graphs, further properties of expansion constants
   Additional material: Alternative expansion constant (3.5) and comparison (Lemma 3.1.15), further example computations of expansion constants

5. Digression: Finite Markov chains ([LPW09], Chapter 1.1 – 1.5)
   Finite Markov chains (with examples), irreducibility, periodicity, random walks on (simple, finite) graphs, stationary distribution (existence and uniqueness)

6. Random walks (Sect. 3.2, p. 51–59)
   Random walks on graphs, Random walks on Cayley graphs, Measure and functions on a graph, Markov operator
   Note: This section uses notions like *random variables*, *probability spaces* and L^2-*functions*, but all of these can be phrased also in more elementary terms.

7. Random walks II (Sect. 3.2, p. 59–68)
   Spectral properties of the Markov operator, Corollary 3.2.20, equidistribution radius, examples, convergence to equilibrium, omit Proposition 3.2.31

8. Random walks and expansion (Sect. 3.3, p. 72–80)
   Absolute expanders, Cheeger inequality, Buser inequality (maybe without proof), examples, expander mixing lemma

9. The discrete Laplace operator and expansion of Cayley graphs (Sect. 3.4 and 3.5, p. 80–90)
   Normalized Laplace operator, spectral definition of expanders, reformulations for Cayley graphs, perturbations of Cayley graphs, bounding the spectral gap, examples
   Additional material: Esperantist graphs

10. Existence of expanders I (Sect. 4.1 and 4.2, p. 95–104)
    Random expanders, Ramanujan graphs (focus on ideas of proofs instead of presenting all details)

11. Existence of expanders II (Sect. 4.3 and 4.4, p. 104–116)
    Cayley graphs of finite linear groups, Kazhdan's property (T)
    Note: This section uses language from *representation theory* and *topological groups* – nothing too deep, but the relevant concepts should be recalled.

12. Applications of expanders I (Sect. 5.2, p. 119–125)
    Deterministic and probabilistic algorithms, Solovay-Strassen primality test, error reduction using expanders
    Note: This section uses some informal notions from *computer science*.

13. Applications of expanders II

14. Applications of expanders III
    The topics presented in the last two talks can be chosen from the following list (or from suggestions by participants). Where it makes sense, multiple topics can be discussed:

    - The Barzdin-Kolmogorov graph-embedding theorem (Sect. 5.1, p. 117–118)
    - Error correcting codes ([Lub12, Section 3.1])
    - Sieve methods (Sect. 5.3, p. 125–133)
    - Lower bounds on gonalities of Riemann surfaces (Sect. 5.4, p. 134–144)
    - Measuring subsets of finitely generated groups ([Lub12, Section 5.1])
    - Probabilistically checkable proofs and the PCP theorem ([RS07])

In addition, since there are more people interested in the seminar than slots for talks, we might have one extra meeting, at a time to be determined (not in the usual slot on Thursday):

15. Exercise class
    Before the class, the speaker compiles a list of exercises from [Kow] which have not been discussed during the talks. Then in class, the speaker discusses these problems with the participants and presents the solutions.

While it is not mandatory to attend this exercise session or prepare the exercises, it might be very helpful for following the lecture and for understanding the material better.

References

[Kow] E. Kowalski. An introduction to expander graphs. Lecture notes. URL: https://people.math.ethz.ch/~kowalski/expander-graphs.pdf.

[LPW09] D. Levin, Y. Peres, and E. Wilmer. Markov chains and mixing times. *A. M. S*, 2009. URL: https://pages.uoregon.edu/dlevin/MARKOV/markovmixing.pdf.

[Lub12] A. Lubotzky. Expander graphs in pure and applied mathematics. *Bulletin AMS 49*, 2012. URL: https://arxiv.org/abs/1105.2389.

[RS07] Jaikumar Radhakrishnan and Madhu Sudan. On Dinur's proof of the PCP theorem. *Bull. Amer. Math. Soc. (N.S.)*, 44(1):19–61, 2007. doi:10.1090/S0273-0979-06-01143-8.