Martin Aigner · Günter M. Ziegler

Proofs from THE BOOK

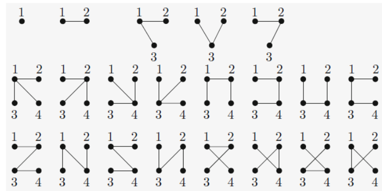
45 chapters

Quadratic reciprocity

$$\left(\frac{q}{p}\right)\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$$

Borromean rings

Cayley's formula



Goals

- practise giving mathematical talks
- see some nice proofs from different areas of mathematics
- get some experience with LaTeX when writing the summary

"The Book"

Typical seminar meeting

Two student talks per meeting
→ 40 min presentation
→ 5 min for questions (& feedback)

Talks on two different chapters
or on one chapter prepared together by the students

Feedback: you have the choice
→ only private feedback by me
→ also short discussion round at the end of the talk

Written summary of your talk

→ prepared in LaTeX
→ Deadline: June 2nd (last day of lectures)

Plan for the semester

Feb 20^{th} Introduction, assignment of talks
Feb 27^{th} Break
Mar 6^{th} – Apr 24^{th} Student talks
May 8^{th}, 15^{th}, 22^{nd} Short talks (?)

short proofs

introduction to LaTeX

Topic	Date	Student
Chapter 3: Binomial coefficients are (almost) never powers Chapter 9: Four times $\pi^2/6$	06.03	Sibylle Schalbetter Laura Depedrini
Chapter 36: Completing Latin squares Chapter 2: Bertrand's postulate	13.03	Annina Eggenschwiler Keerthiga Rajakumar
Chapter 12: The slope problem Chapter 31: Shuffling cards	20.03	Andrin Schneeberger Christian Farkas
Chapter 40: How to guard a museum Chapter 41: Turán's graph theorem	27.03	Jeremy Suter Ira Griesshammer
Chapter 15: The Borromean rings don't exist	03.04	Irena Syla Joel Zülle
Chapter 21: The fundamental theorem of algebra Chapter 44: Of friends and politicians	17.04	Teodora Krstic Alicia Brockmöller
Chapter 19: Sets, functions, and the continuum hypothesis Chapter 6: Every finite division ring is a field	24.04	Philine Schönenberger Ida Krinn

How to give a seminar talk

Below I go through some of the tips from the guide I sent you, and try to illustrate them with examples from The Book of Proofs.

Preparation

For the chapter you speak about:

→ read carefully

→ understand it as thoroughly as possible

Note almost always necessary to add details (for your understanding & your talk)

Exa Chapter 1 gives six proofs that there are infinitely many primes

■ **Second Proof.** Let us first look at the *Fermat numbers* $F_n = 2^{2^n} + 1$ for $n = 0, 1, 2, \ldots$. We will show that any two Fermat numbers are relatively prime; hence there must be infinitely many primes.

$$\begin{aligned} F_0 &= 3 \\ F_1 &= 5 \\ F_2 &= 17 \\ F_3 &= 257 \\ F_4 &= 65537 \\ F_5 &= 641 \cdot 6700417 \end{aligned}$$

The first few Fermat numbers

The logic in the last sentence looks plausible, but we can unpack to make clearer!

Def Integers a, b are relatively prime if the only positive integer dividing a and b is 1.
$\Longleftrightarrow$ $\nexists$ prime p such that $p \mid a$ and $p \mid b$

Claim 1 F_n and F_m are relatively prime for $m \neq n$

Assuming Claim 1, we prove that $\exists$ infin. many primes:

From formula:

$F_n = 2^{2^n} + 1 > 1 \Rightarrow$ F_n has some prime factor p_n

The numbers p_i must be pairwise distinct ($p_n = p_m \Rightarrow F_n, F_m$ not coprime ↯ Claim 1)

$\Rightarrow \{p_1, p_2, p_3, \ldots\}$ is infinite set of primes.

We see: 2 lines in book $\rightsquigarrow$ 10 lines when recalling def. & adding details

Once logic is clear, think about what to write on board / slides:

$\nexists$ prime p: $p \mid F_n$ and $p \mid F_m$

Claim 1 F_n and F_m are relatively prime for $m \neq n$

Now: Claim 1 $\Rightarrow \exists$ infin. many primes:

$F_n = 2^{2^n} + 1 > 1 \Rightarrow$ F_n has some prime factor p_n

The numbers p_i must be pairwise distinct ($p_n = p_m \Rightarrow F_n, F_m$ not coprime $\lightning$ Claim 1)

$\Rightarrow \{p_1, p_2, p_3, \dots\}$ is infinite set of primes. ⁂

Motivation management

Above the concept of "Fermat numbers" came out of nowhere. If this happens too many times, it's easy to lose your audience!

E.g. look at the following (see PfTB, p.7)

Let $S = (s_1, s_2, s_3, \dots)$ be a sequence of integers. We say that

- S is *almost injective* if every value occurs at most c times for some constant c,
- S is *of subexponential growth* if $|s_n| \leq 2^{2^{f(n)}}$ for all n, where $f\colon \mathbb{N} \to \mathbb{R}_+$ is a function with $\frac{f(n)}{\log_2 n} \to 0$.

Theorem. *If the sequence $S = (s_1, s_2, s_3, \dots)$ is almost injective and of subexponential growth, then the set $\mathbb{P}_S$ of primes that divide some member of S is infinite.*

($\rightsquigarrow$ in fact: PfTB does give motivation before)

How could we have motivated the Fermat numbers?

Example to make formula less mysterious

Consider the Fermat numbers $F_n = 2^{2^n} + 1$ for $n = 0, 1, \dots$

<u>Exa</u> $F_0 = 2^1 + 1 = 3$

$F_1 = 2^2 + 1 = 5$

$F_2 = 2^4 + 1 = 17$

$\vdots$

Interaction with audience

<u>Interesting history</u>

<u>Conjecture</u> (Fermat)

F_n prime $\forall n$.

relate to bigger goal

↳ would finish our proof that $\exists$ inf. many primes

amusing anecdote

<u>Problem</u> Conjecture is (very) wrong!

$F_5 = 2^{32} + 1 = 641 \cdot 6700417$

<u>New conjecture</u>

$F_1, \dots, F_4$ are <u>only</u> F_n which are prime.

preview of following part

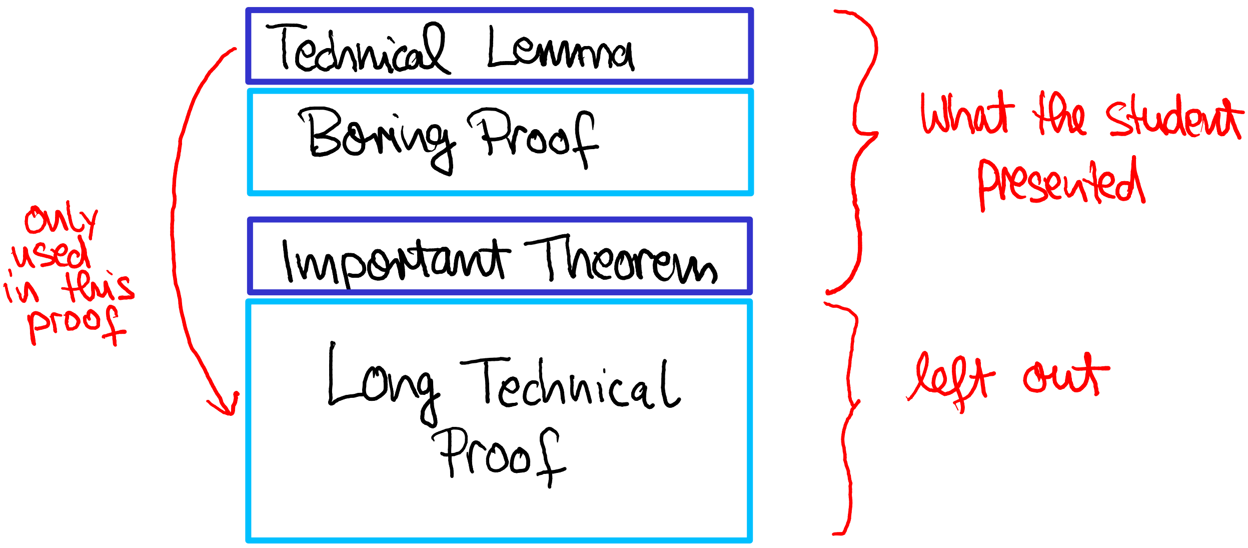
Still we can use the F_n to find infinitely many primes: ...

Time management

40 min for your talk ~> make the most of it!

- Easy to understand → try to do quickly
  Exa Motivation on Fermat numbers
  How Prepare slides / hidden board, only say in words, don't write whole sentences

- Boring or unnecessary → try to leave out
  Exa Student in earlier seminar had to talk about a chapter w/ following structure:



~> can leave out Technical Lemma, more time for Theorem and Applications.

- Good way to manage time: test/practice talks
  → find out how long different parts take
  → Spot mistakes in notes
  → helps against nervousness

Exa Before test talk: $F_1 = 2^1 + 1 = 3$

Questions

Standard situation in seminars: no one asks questions!
I very much hope that we can change this here:

advantages ↗ audience: resolve confusion, stay focused
↘ speaker: practice!

Also: speaker can always pass on question to me.

Feedback

You can choose to have open feedback round after talk

advantages ↗ audience: start noticing what works/doesn't work
↘ speaker: actual feedback from target audience

Summary

→ understand text as well as possible, add details
→ manage motivation, e.g. by explaining what you do and why you do it
→ manage time by doing simple things quickly, leaving unnecessary parts out
→ questions and feedback are good for audience & speaker

Thanks for listening!

Questions?

Feedback?