MAT773.1 Topics in convex geometry (HS22)

Johannes Schmitt

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The main reference for the seminar will be the book [Zie95] by Ziegler. Below are the outlines of the topics of individual talks and the concepts and results that should be explained. Depending on the amount of material in the sections below (which refer to [Zie95] unless stated otherwise), it might be necessary to either shorten the presented material (e.g. by leaving out technical details in proofs) or to add more material (e.g. by discussing some of the exercises (which appear at the end of the chapter), working out some additional example or filling in details of arguments).

1. Basic definitions (p. 2–10)
   affine maps and subspaces, convexity and convex hull, polytopes (with first examples), simplicial polytopes, pyramids and products
   Compared to the treatment in [Zie95], it would be good to add some examples to many of the concepts he discusses (e.g. specify an affine subspace of $\mathbb{R}^3$, write it as an affine hull of points, etc)

2. Further examples and main theorems (p. 16 – 21, 27 – 30)
   Example 0.9, permutahedron (but not associahedron and other variants), 0/1-polytopes, traveling salesman polytope (just idea, no technical details); cones, Minkowski sum, stating Theorems 1.1, 1.2, 1.3

3. Proofs of main theorems (p. 30–39)
   Proving Theorem 1.2 from Theorem 1.3, idea of Fourier-Motzkin elimination for polyhedra (Sect. 1.2), Fourier-Motzkin elimination for cones (Sect. 1.3)

4. Farkas lemmas and further basic results (p. 39– 47)
   State the Farkas lemmas, show e.g. proof of I, give examples and

maybe explain intuition for III, IV; lineality space and recession cone, homogenization, Carathéodory's theorem

5. Faces of polytopes (p. 51 – 59)
   Faces, vertices, facets, etc, Proposition 2.2 and Proposition 2.3 with proofs, face figure and Proposition 2.4 without proof, face lattice and Theorem 2.7 (possibly with proof)

6. Polarity and the representation theorem for polytopes (p. 59 – 67)
   Lemma 2.8, (relative) interiors, polar sets, Theorem 2.11 in detail, state remaining results in Section 2.3 (probably without proofs), representation theorem for polytopes, simplicial and simple polytopes

7. Platonic solids
   Describing and classifying the five platonic solids (geometric proof, topological proof), vertex coordinates, symmetry groups, dual/polar polyhedra, appearance in nature, further related topics (uniform polyhedra, regular tessellations, higher dimensions)
   Resources:

   - Wikipedia page and references there
   - Lecture notes by Norman Do
   - Lecture "Symmetry groups of Platonic solids" by John O'Connor
   - Lecture notes by Mikhail Lavrov

   Note: The discussion of symmetry groups uses some notions from the lecture Algebra.

8. Linear programming and related concepts (p. 77–83)
   Lines and linear functions in general position, the graph of a polytope, orientations, linear programming/simplex method for geometers, state the Hirsch Conjecture, give some background for this conjecture and mention the counterexample, which was found by Santos in 2011, *after* the publication of [Zie95]!

9. More on graphs of polytopes (p. 93–96, 103–104, 114)
   Telling a simple polytope from its graph, Balinski's theorem that the graph of a d-polytope is d-connected, state Steinitz' theorem (optional: talk about idea of proof), Corollary 4.9, optional: Theorem 4.11

10. Polyhedral complexes and Schlegel diagrams (p. 127–137)
    Polyhedral complexes, face poset, polytopal subdivision, regular subdivisions, a non-regular subdivision, Schlegel diagrams (with lots of pictures)

11. Review and exercise session
    Review the material from the previous lectures, recalling the main theorems, or explaining some of the examples in more detail; create a short sheet of exercises from the book that is distributed to the participants of the seminar before the lecture, and explain the solutions in class

12. The simplex algorithm
    Explain in detail the simplex algorithm (that was mentioned in lecture 8). Here the focus should be on the concrete matrices, vectors, etc that appear, and the lecture should include a concrete example computation where the algorithm is demonstrated.
    This lecture could follow e.g. the excellent German Wikipedia page of the simplex algorithm.

13. Further algorithms and practical implementations
    This lecture should discuss further algorithmic aspects of convex geometry. On the theoretical side, a possibility is to explain a slow and a fast algorithm for the convex hull in the plane (p. 2 – 8 of [dBCvKO08]). On the practical side (in particular for students who attended the course MAT007 on computer algebra last semester) one could have a computer demonstration of mathematical software for convex geometry (e.g. SageMath, Polymake, Gfan). Possible topics for the demonstration are: how to enter a polytope/polyhedron via vertices+cones or inequalities, showing plots of interesting polyhedra, computing symmetry groups, face lattices, etc.

14. Ehrhart theory (p. 62–73 of Lecture Notes "Lattice Polytopes")
    Lattice polytopes, Ehrhart counting function (with some examples), state Ehrhart's Theorem (Theorem 3.14), give ingredients and idea of proof (involving generating series), state Ehrhart reciprocity (Theorem 3.43) and give examples
    Further resources

    - Wikipedia page

- Section 4.6.2 of [Sta12]
- Youtube talk
- Blog post by Nicolai Hähnle

References

[dBCvKO08] Mark de Berg, Otfried Cheong, Marc van Kreveld, and Mark Overmars. *Computational geometry*. Springer-Verlag, Berlin, third edition, 2008. Algorithms and applications. `doi:10.1007/978-3-540-77974-2`.

[Sta12] Richard P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.

[Zie95] Günter M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. `doi:10.1007/978-1-4613-8431-1`.